# 104. The Cauchy Problem for a Class of Hyperbolic Operators with Triple Characteristics

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**1. Introduction.** In the $C^\infty$ category the well-posedness of the Cauchy problem for hyperbolic operators depends in general on the behaviour of the lower order terms. When the characteristic roots are at most double, necessary and (almost) sufficient conditions have been given by Ivrii-Petkov [5], Ivrii [4] and Hörmander [2]. For higher order multiplicities results on the (microlocal) Cauchy problem have been proved by Bernardi [1] in the involutive case and by Nishitani [6] in the "effectively" hyperbolic case. In comparison with these last two cases the Levi conditions in a non-involutive and "non-effectively" hyperbolic situation seem to be much more involved and that is the reason why we restricted ourselves to multiplicity of order three.

Let us now introduce our notations. Let $\Omega\subset \boldsymbol{R}^{n+1}$ an open subset, $x=(x_0, x_1, \cdots, x_n)\in\Omega$, $D_{x_j}=(1/i)\partial_{x_j}$, $j=0, \cdots, n$, $x=(x_0, x')$. Let $P(x, D_x)=P_m(x, D_x)+P_{m-1}(x, D_x)+\cdots$, be a hyperbolic differential operator of order $m$. We denote by $\Gamma_\rho$, $\rho\in\Omega\times\boldsymbol{R}^n\backslash\{0\}$, the hyperbolicity cone of $P$ in $\rho$ and by $\Gamma_\rho^\sigma$ the polar of $\Gamma$ with respect to the symplectic two-form $\sigma=d\xi\wedge dx=d\omega$, $\omega$ the canonical one-form. See [7] for the definition of $\Gamma_\rho$. We recall the definition of the subprincipal symbol of $P: P^s_{m-1}(x, \xi)=P_{m-1}(x, \xi)+(1/2)\sum_{j=0}^n \partial^2_{x_j\xi_j} p_m(x, \xi)$. It is invariantly defined at double characteristic points of $p_m$. If $q$ is a hyperbolic operator with double characteristics we note by $F_q$ the fundamental matrix of $q_2$, the principal symbol of $q$, and by $\mathrm{Tr}^+F_q=\sum\lambda_j$, where $\pm i\lambda_j\in sp(F_q)$. See [3] for precise definitions. We now state our results:

**2. Results.** We shall make the following assumptions on $P$.

H1) The principal symbol of $P$, $p_m(x, \xi)$ is hyperbolic with respect to $\xi_0$.

H2) The characteristic roots of $\xi_0\to P_m(x, \xi_0, \xi')$ have multiplicities at most of order 3 and the triple characteristic set $\Sigma=\{(x, \xi)\in\Omega\times\boldsymbol{R}^n\backslash\{0\}\mid p_m(x, \xi)=dp_m(x, \xi)=d^2p_m(x, \xi)=0\}$ is a $C^\infty$ manifold such that rank $\sigma|_\Sigma=$ const and $\omega$ does not vanish identically on $T\Sigma$.

Let $\rho\in\Sigma$:

H3)$_\rho$ Denote by $\mathrm{T}_\rho(\Omega\times\boldsymbol{R}^n\backslash\{0\})\ni\delta z\to P_{m,\rho}(\delta z)$ the localization of $P_m$ at $\rho$ (see e.g. [7]). Then

(i) $P_{m,\rho}(\delta z)=L_1(\delta z)Q_2(\delta z)$ where $L_1(\delta z)=\delta\xi_0-l_1(\delta x, \delta\xi')$, $l_1$ being a real linear form in $(\delta x, \delta\xi')$.

(ii) $Q_2(\delta z)$ is a real hyperbolic quadratic form such that:

a) $\dim \operatorname{Ker} F_{Q_2} = \dim T_\rho \Sigma$

b) $\operatorname{Ker} F_{Q_2}^2 \cap \operatorname{Im} F_{Q_2}^2 = \{0\}$

c) $\forall v \neq 0,\ v \in V^+ = \bigoplus_{\substack{i\lambda \in sp(F_{Q_2}) \\ \lambda > 0}} \operatorname{Ker}(F_{Q_2} - i\lambda I), \quad -i\sigma(v, \bar{v}) > 0$

d) $sp(F_{Q_2}) \subset i\boldsymbol{R}$.

$\mathrm{H4})_{\rho,l}$ $\qquad H_{L_1} \in \Gamma^\sigma_{Q_2} \cap \operatorname{Ker} F_{Q_2}$

$\mathrm{H4})_{\rho,s}$ $\qquad H_{L_1} \in \operatorname{Int}(\Gamma^\sigma_{Q_2}) \cap T_\rho \Sigma$.

Set $\Omega_t = \{x \in \Omega \mid x_0 < t\}$. For a definition of correctly posed Cauchy problem for $P$ in $\Omega_t$ we refer to Hörmander, [2].

Our first result will be the following:

**Theorem 1.** Assume that the Cauchy problem of $P$ is well posed in $\Omega_t$, $t$ small. Let $\rho$ be a triple characteristic point of $P_m$. If H1), $\mathrm{H3})_\rho$ (i), (ii), b)–d) and $\mathrm{H4})_{\rho,l}$ hold, then the following conditions are necessary:

$\mathrm{L1})_l$ $\qquad \operatorname{Re} p^s_{m-1}(\rho) = 0, \qquad H_{\mathrm{Tr}+F_{Q_2 L_1} \pm \operatorname{Re} p^s_{m-1}(\rho)} \in \Gamma^\sigma_\rho$.

L2) $\qquad \operatorname{Im} p^s_{m-1}(\rho) = 0, \qquad H_{\operatorname{Im} p^s_{m-1}}(\rho) = 0$.

Next we have

**Theorem 2.** Assume that $P$ verifies hypotheses H1), H2), and $\forall \rho \in \Sigma$ $\mathrm{H3})_\rho$, $\mathrm{H4})_{\rho,s}$. Then if:

$\mathrm{L1})_s$ $\qquad \operatorname{Re} p^s_{m-1}(\rho) = 0, \qquad \forall \rho \in \Sigma$

$\qquad H_{\mathrm{Tr}+F_{Q_2 L_1} \pm \operatorname{Re} p^s_{m-1}(\rho)} \in \operatorname{Int}(\Gamma^\sigma_\rho), \qquad \forall \rho \in \Sigma$.

and L2) hold, the Cauchy problem for $P$ in $\Omega_0$ is well posed.

**Remarks.** (i) As a consequence of $\mathrm{H4})_{\rho,s}$ we have that $P_{m,\rho}$ is a strictly hyperbolic polynomial; moreover outside $\Sigma$ $p_m$ has only simple characteristics and the Hamiltonian flow of $p_m$ does not have limit points on $\Sigma$.

(ii) As a matter of fact Theorem 1 is true in larger assumptions: in particular we can drop hypotheses $\mathrm{H3})_\rho$ (ii), b), c).

(iii) If $P = L \cdot B$, where $L$ is a first order differential hyperbolic operator and $B$ is a second order non effectively hyperbolic operator with double characteristics, our Levi conditions L1), L2) give back the usual Ivrii-Petkov-Hörmander conditions for $B$. Moreover the known Ivrii-Petkov necessary conditions are implied by ours.

(iv) All the assumptions and the Levi conditions L1), L2) are invariant under canonical transformations.

Let us finally give an example of an operator satisfying our conditions:

$$P(x, D) = (D_0 - lD_1)(-D_0^2 + D_1^2 + D_2^2 + x_2^2 D_n^2) + (c_0 D_0 + c_1 D_1 + c_2 D_2) D_n.$$

In this case $\mathrm{H4})_l$ says $|l| \leq 1$. L2) states that $c_j \in \boldsymbol{R}$, $j = 0, 1, 2$. Finally $\mathrm{L1})_l$ is equivalent to:

$$1 - |c_0| \geq \sqrt{c_2^2 + (|c_1| - l)^2}$$

and these are the necessary and sufficient conditions in order that the Cauchy problem for $P$ be well-posed.

## References

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