100. A Cohomological Construction of Swan Representation over the Witt Ring. II

By Osamu HYODO
Department of Mathematics, Nara Women's University

(Communicated by Shokichi IYANAGA, M. J. A., Nov. 14, 1988)

This is continued from [0].

3. In this section we give the construction of Swan representations. Let $K=k((T^{-1}))$ be a complete discrete valuation field and M be a finite Galois extension of K with Galois group G. N. M. Katz proved the following

Theorem ([7] Theorem (1.4.1)). *There exists a canonical finite etale Galois covering*

$$U \longrightarrow \boldsymbol{G}_{m,k}=\operatorname{Spec} k[T, T^{-1}]$$

which satisfies the following properties.

(1) $$U\otimes_{G_{m,k}} \operatorname{Spec} k((T^{-1}))\simeq \operatorname{Spec} M$$

(2) $U\otimes_{G_{m,k}} \operatorname{Spec} k((T))$ *is a disjoint union of the spectra of tamely ramified extensions of* $k((T))$.

We denote by $X\xrightarrow{g}\boldsymbol{P}^1_k$ the compactification of $U\longrightarrow \boldsymbol{G}_{m,k}$. Note that g factors as $X\longrightarrow \boldsymbol{P}^1_m \longrightarrow \boldsymbol{P}^1_k$, where m denotes the residue field of M. We denote by D_0 (resp. D_∞) the inverse image of $T=0$ (resp. $T=\infty$) with reduced scheme structure. Then $X\backslash U=D_0 \coprod D_\infty$. Let $W\Omega^{\cdot}_X(\log D_0-\log D_\infty)$ be the de Rham-Witt complex with logarithmic poles along D_0 and with minus logarithmic poles along D_∞ as in §1. As D_0 and D_∞ are stable under the action of G, $\sigma\in G=\operatorname{Gal}(U/\boldsymbol{G}_{m,k})$ acts on the free W-module

$$H^1(X, W\Omega^{\cdot}_X(\log D_0-\log D_\infty))$$

by transportation of structures. The following Proposition shows that this is the desired space of the Swan representation of G.

Proposition. *The trace of the action of* $\sigma\in G$ *on*

$$H^1(X, W\Omega^{\cdot}_X(\log D_0-\log D_\infty))$$

coincides with $Sw_G(\sigma)$.

In the following we denote the alternating sum of the trace of the action of σ on free $W(k)$-modules by

$$\operatorname{Tr}(\sigma): R\Gamma(X,\ \ )) := \sum_{q\geq 0} (-1)^q \operatorname{Tr}(\sigma: H^q(X,\ \ )).$$

By Lemma and exact sequences (∗∗) in §1, it suffices to show

$$\operatorname{Tr}(\sigma: R\Gamma(X, W\Omega^{\cdot}_X)=\begin{cases} d^\sigma+(-Sw_G(\sigma)+f) & \text{for } \sigma\in I,\\ 0 & \text{for } \sigma\notin I,\end{cases}$$

where d^σ denotes the degree of the closed subscheme of D_0 fixed by σ and $f=[m:k]$ coincides with degree of D_∞ over k.

The proof of this formula is the same as the proof of the Weil formula [4] §5: The case $\sigma=1$ is the Hurwitz formula. The case $\sigma\neq 1$ is deduced from the fixed point formula (crystalline cohomology is a Weil cohomology theory [3]). We omit the detail.

Remark and question. (1) Contrary to the l-adic case, $Sw_{G,p}$ can not always be realized as a projective $W(k)[G]$-module. This phenomenon seems to suggest that one can not expect the "Grothendieck-Ogg-Shafarevich formula" for crystals defined over open smooth curves. (cf. [7] § (1.6).)

(2) Nevertheless, are there nice theory of the Swan conductor (or irregularity) for crystals?

Reference*)

[0] O. Hyodo: A cohomological construction of Swan representation over the Witt ring. I. Proc. Japan Acad., **64A**, 300–303 (1988).

*) [1]–[8] as in [0].