91. A Holomorphic Structure of the Arithmetic-geometric Mean of Gauss

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§1. **Introduction.** For a, $b>0$, we define two sequences $\{a_n\}$ and $\{b_n\}$ by

$$a_0=a, \qquad b_0=b$$
$$a_{n+1}=\frac{1}{2}(a_n+b_n), \quad b_{n+1}=\sqrt{a_n b_n}, \quad n=0,1,2,\cdots. \tag{1.1}$$

It is well known and easily proved that both sequences converge to a common limit

$$M(a,b)=\lim a_n=\lim b_n,$$

which is called the arithmetic-geometric mean of a and b.

When a and b are complex numbers, we can define a sequence $\{(a_n, b_n)\}$ by the same algorithm (1.1). However, since there are two choices for b_{n+1} at each step of (1.1), we get uncountably many sequences $\{(a_n, b_n)\}$, which make the situation much more complicated than in the real case. Although the study of this case was initiated by Gauss, we refer to Cox [1,2] as a modern account of what happens to the arithmetic-geometric mean of two complex numbers.

We assume

(A) $\qquad a, b\in \boldsymbol{C}, \quad ab\neq 0 \quad \text{and} \quad a\pm b\neq 0.$

The excluded cases, though trivial, will turn out to be singular in a certain sense. It is easy to see that a_n and b_n also satisfy (A) for all $n\geq 0$.

A pair (a_n, b_n) is called *the right choice* if

$$\mathrm{Re}\,(b_n/a_n)>0 \quad \text{or} \quad \mathrm{Re}\,(b_n/a_n)=0, \quad \mathrm{Im}\,(b_n/a_n)>0.$$

Note that one of (a_n, b_n) and $(a_n, -b_n)$ is always the right choice, while the other is "the wrong choice".

One can prove that for any sequence $\{(a_n, b_n)\}$ the limit $\tau=\lim a_n=\lim b_n$ exists and that $\tau\neq 0$ if and only if all but finitely many of (a_n, b_n) are right choices ([1], [3]). Let $\mathfrak{M}(a,b)$ denote the set of such non-zero limits and $M(a,b)$ denote the limit attained by $\{(a_n, b_n)\}$ where (a_n, b_n) is the right choice for all $n\geq 1$.

Theorem (Cox [1], Geppert [4]). *Let a and b satisfy* (A). *Then all the values τ of $\mathfrak{M}(a,b)$ are given by*

$$\tau^{-1}=pM(a,b)^{-1}+iqM(a+b, a-b)^{-1},$$

where p and q are arbitrary relatively prime integers satisfying $p\equiv 1 \bmod 4$ and $q\equiv 0 \bmod 4$.

The purpose of this note is to give a sketch of a proof different from

Cox's; our proof does not rely on theta identities, but on certain integrals on the elliptic curve, $y^2=x(1-x)(a^2(1-x)+b^2x)$:

$$(1.2)\qquad \begin{aligned} M(a,b)^{-1}&=\frac{1}{\pi}\int_0^1\frac{dx}{y},\\ iM(a+b,a-b)^{-1}&=\frac{1}{\pi}\int_0^{-\infty}\frac{dx}{y}. \end{aligned}$$

The first formula is introduced in [1] in a slightly different fashion. The second follows from the first by a change of the variable: $(1-x)(1-x')=1$.

§2. **Connectedness of $\mathfrak{M}(z)$.** Due to the homogeneity, $M(\lambda a,\lambda b)=\lambda M(a,b)$, $\mathfrak{M}(\lambda a,\lambda a)=\lambda\mathfrak{M}(a,b)$, $\lambda\in\boldsymbol{C}$, we may put $a=1$, $b=z$ and write $M(z)=M(1,z)$ and $\mathfrak{M}(z)=\mathfrak{M}(1,z)$. The assumption (A) is now

$$z\in\boldsymbol{C}_0:=\boldsymbol{C}\backslash\{0,\pm 1\}.$$

$a_n(z)$ and $b_n(z)$ are algebraic functions possibly with branch singularities at 0, ± 1 and ∞. $\mathfrak{M}(z)$ consists of values of holomorphic functions; this follows from the fact that $\lim a_n(z)=\lim b_n(z)$ locally defines a holomorphic function.

The first part of our proof consists in showing that, for any fixed $z_0\in\boldsymbol{C}_0$,

$$(2.1)\qquad \mathfrak{M}(z_0)=\{\gamma_*M(z_0)\,;\,[\gamma]\in\pi_1(\boldsymbol{C}_0\,;z_0)\},$$

where γ_*f denotes the holomorphic function obtained by the analytic continuation of f along the path γ. The above statement is an easy consequence of the following observation.

Lemma. *Let $z_0\in\boldsymbol{C}_0$ and $\{(a_n(z_0),b_n(z_0))\}_{n=0}^{\infty}$ be a sequence defined by the algorithm (1.1) with $a_0=1$ and $b_0=z_0$. Suppose that there is a number $N(\geq 2)$ such that (a_n,b_n) is the right choice for all $n\geq N$. Then there exists a point z_1 and a curve γ in $\boldsymbol{C}_0$ connecting z_0 to z_1 such that $(\gamma_*a_n(z_1),\gamma_*b_n(z_1))$ is the right choice for every $n\geq N-1$.*

§3. **A monodromy representation.** (2.1) says that all the values of $\mathfrak{M}(z_0)$ are attained by the analytic continuation of $M(z)$ along various cycles of $\pi_1(\boldsymbol{C}_0\,;z_0)$. We will now study $\gamma_*M(z_0)$ when $z_0=1/2$; the general case follows easily from this if we connect z_0 to $1/2$ by a suitable path.

Let γ_1 be the circle of radius $1/2$ around the center $z=1$ and γ_0 the circle of radius $1/2$ around $z=0$; both are oriented in the positive direction. We will consider them as elements of $\pi_1(\boldsymbol{C}_0\,;1/2)$. Let γ_{-1} be the cycle that starts at the point $1/2$, moves along the upper semi-circle of γ_0, then goes on the circle of radius $1/2$ around the point -1 and finally returns to the point $1/2$ traveling the same upper half of γ_0. Note that $\pi_1(\boldsymbol{C}_0\,;1/2)$ is a free group generated by γ_{-1}, γ_0 and γ_1.

We now write (1.2) in the following form:

$$(M(z)^{-1},\ iM(1+z,1-z)^{-1})=(\sqrt{\lambda}/\pi)(u_1(\lambda),u_2(\lambda)),$$

where $\lambda=\lambda(z)=(1-z^2)^{-1}$ and

$$u_1(\lambda)=\int_0^1\frac{dx}{y(\lambda)},\qquad u_2(\lambda)=\int_0^{-\infty}\frac{dx}{y(\lambda)}$$

with $y(\lambda)^2=x(1-x)(\lambda-x)$.

The map $\lambda(z): C_0 \to C_1 := C \setminus \{0,1\}$ induces the map $\lambda_*: \pi_1(C_0; 1/2) \to \pi_1(C_1; 4/3)$. We then have

$$\lambda_* \gamma_0 = \delta_1^2, \quad \lambda_* \gamma_1 = \delta_\infty^{-1}, \quad \lambda_* \gamma_{-1} = \delta_1^{-1} \delta_\infty^{-1} \delta_1,$$

where δ_1 and δ_∞ are cycles $\in \pi_1(C_1; 4/3)$ defined as follows: δ_1 moves once around the point 1 (but not 0) and δ_∞ moves once around the points 0 and 1, both in the positive direction.

We are now concerned with what happens to u_1 and u_2 when λ moves along the cycle δ_1 or δ_∞. This actually corresponds to the question of a monodromy representation of a Legendre equation,

$$\lambda(\lambda-1)u'' + (2\lambda-1)u' + (1/4)u = 0,$$

since u_1 and u_2 form a fundamental system of the equation. However, we do not need this fact here. A continuous variation of the paths of integration for u_1 and u_2 in accordance with the move of λ leads to

$$\delta_{1*}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = U^{-1}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

$$\delta_{\infty*}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = V\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Therefore, all

$$\gamma_*(M(z)^{-1}, iM(1+z, 1-z)^{-1}), \qquad \gamma \in \pi_1\left(C_0; \frac{1}{2}\right)$$

are obtained by the action of the subgroup Γ (of $SL_2(Z)$) generated by

$$U^2, V \quad \text{and} \quad U^{-1}VU.$$

Now, we define $\Gamma_2(4)$ as the group of matrices

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \quad \text{of} \quad SL_2(Z)$$

such that $p \equiv s \equiv 1 \pmod 4$, $q \equiv 0 \pmod 4$ and $r \equiv 0 \pmod 2$. The last part of our proof is devoted to proving $\Gamma = \Gamma_2(4)$. Our theorem is an immediate consequence of this, since the set of the first rows of the matrices of $\Gamma_2(4)$ equals

$$\{(p, q); p \text{ and } q \text{ are relatively prime, } p \equiv 1 \pmod 4 \text{ and } q \equiv 0 \pmod 4\}.$$

References

[1] Cox, D.: The arithmetic-geometric mean of Gauss. L'Enseignement Math., **30**, 275–330 (1984).

[2] ——: Gauss and the arithmetic-geometric mean. Notices of the AMS, **32**, 147–151 (1985).

[3] von David, L.: Arithmetisch-geometrisches Mittel und Modulfunktion. J. Reine Angew. Math., **159**, 154–170 (1928).

[4] Geppert, H.: Zur Theorie des arithmetisch-geometrischen Mittels. Math. Ann., **99**, 162–180 (1928).