89. *Decreasing Streamlines of Solutions and Spectral Properties of Linearized Operators for Semilinear Elliptic Equations*

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§1. Introduction. Let $\Omega\subset \boldsymbol{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$ and $f: \boldsymbol{R}\to\boldsymbol{R}$ be a C^1 function. We consider the semilinear elliptic equation

(1) $$-\Delta u=f(u),\quad u>0 \text{ (in } \Omega),\quad u=0 \text{ (on } \partial\Omega).$$

Then the linearized operator around the solution $u=u(x)\in C^2(\Omega)\cap C^0(\bar{\Omega})$ is given by $A\equiv A(u)=-\Delta-f'(u)$ in $L^2(\Omega)$ with $D(A)=H_0^1\cap H^2(\Omega)$. In the previous work [1], we have given some streamlines in Ω along which the solution decreases, when Ω is a symmetric domain in $\boldsymbol{R}^2$. There, we restricted ourselves to the mild solution, that is, the case when the second eigenvalue $\mu_2=\mu_2(u)$ of $A=A(u)$ is positive. In this article, we shall note that conversely, the decreasing streamlines of the solution contain some information about the eigenvalues of $A(u)$.

Thus, we suppose that the domain is the unit ball: $\Omega=\{|x|<1\}\subset\boldsymbol{R}^N$. Then from [5], every solution $u=u(x)$ of (1) is radial: $u=u(|x|)$ and $u_r<0$ for $0<r=|x|<1$. Therefore, the set of eigenvalues $\sigma(u)$ of $A(u)$ is divided as $\sigma(u)=\cup_{m=0}^{\infty}\sigma_m(u)$ according to the principle of separation of variables. Namely, let $\{\rho_m\}_{m=0}^{\infty}$ $(0=\rho_0<\rho_1<\rho_2<\cdots)$ be the eigenvalues of $-\Lambda$, where Λ denotes the Laplace-Beltrami operator on $S^{N-1}=\{|x|=1\}$. In fact we have $\rho_m=m(2\nu+m)$, where $2\nu=N-2$. Further, multiplicity κ_m of ρ_m (and hence that of $\mu\in\sigma_m(u)$) is as follows: for $N=2$ we have $\kappa_m=1$ $(m=0)$ and $\kappa_m=2$ $(m\geqq 1)$; for $N>2$ we have $\kappa_m=(2m+N-2)(m+N-3)!/(N-2)!\,m!$ (see, e.g. [9]). Then $\sigma_m(u)$ denotes the set of eigenvalues of the ordinary differential operator $A_m(u)=-(d^2/dr^2)-((N-1)/r)(d/dr)-c(r)+(\rho_m/r^2)$ with $(d/dr)\cdot|_{r=0}=\cdot|_{r=1}=0$, where $c(r)=f'(u)$.

Now, for these sets $\sigma_m(u)$ $(m=0,1,2,\cdots)$, we claim the following, where $\boldsymbol{R}_+=(0,\infty)$:

Theorem. *If* $f(\boldsymbol{R}_+)\subset\boldsymbol{R}_+$, *then* $\sigma_m(u)\cap(-\infty,0]=\phi$ *for* $m\geqq 1$. *In particular,* $\dim\operatorname{Ker} A(u)$ *is at most* 1 *for any solution u on the ball* $\Omega=\{|x|<1\}\subset\boldsymbol{R}^N$, *provided that* $f(\boldsymbol{R}_+)\subset\boldsymbol{R}_+$.

§2. Proof of Theorem. Set $\sigma_m(u)=\{\mu_j^m\,|\,j=0,1,2,\cdots\}$ with $-\infty<\mu_0^m<\mu_1^m<\cdots$. Since $\rho_{m'}>\rho_m$ if $m'>m$, we have $\mu_0^1<\mu_0^2<\cdots$ and hence we have only to prove that $\mu_0^1>0$.

The eigenfunction φ of $A(u)$ corresponding to μ_0^1 is of the form $\varphi(x)=$

$\Phi(r)\chi(\omega)$ $(x=r\omega)$, where $\Phi=\Phi(r)$ denotes the eigenfunction of $A_1(u)$ and $\chi=\chi(\omega)$ denotes the second eigenfunction of $-\Delta$. Thus $\chi=\chi(\omega)$ has exactly two nodal domains $S_\pm$ on $S^{N-1}=\{|x|=1\}$ and $S_\pm$ are chemi-spheres. Therefore, $\mu_0^1>0$ follows from the existence of a $w\in C^2(\Omega_+)\cap C^0(\bar{\Omega}_+)$ satisfying

(2) $\quad (-\Delta-f'(u))w<0,\quad w<0 \text{ (in } \Omega_+) \quad \text{and} \quad w=0 \text{ (on } \partial\Omega_+)$

by Jacobi's method ([2]), where $\Omega_+=\{x\in\Omega \mid x_n>0\}$ denotes a chemi-ball. We shall give such a w for the cases of $N=2$ and $N=3$, for simplicity.

The case $N=2$. Let $[\ \ ]: C\to \boldsymbol{R}^2$ be the canonical mapping. Through some calculations we can derive from $-\Delta u=f(u)$ that

(3) $\quad -\Delta w=2(\operatorname{Re}\nu_z)f(u)+f'(u)w \quad (\text{in } \Omega),$

where $w=\nabla u\cdot[\nu]$ for each holomorphic function $\nu=\nu(z)$ $(z=x_1+ix_2)$. Taking $\nu(z)=i(1+z^2)$, we have $\operatorname{Re}\nu_z=-2x_2<0$ in $\Omega_+=\{|x|<1,\ x_2>0\}$, so that $(-\Delta-f'(u))w<0$ in Ω_+ by $f(u)>0$. Further, each flow l starting from $\bar{\Omega}_+\cap\{x_2=0\}$ subject to the vector field $\nu=\nu(z)$ goes outside from the level set $\{|x|=c\}$ $(0<c<1)$ in Ω_+. Hence $w<0$ in Ω_+, because $u=u(|x|)$ with $u_r<0$ $(0<r<1)$. Finally l is orthogonal to $\{x_2=0\}$ and goes along $\partial\Omega$ if it starts from the end points of $\bar{\Omega}_+\cap\{x_2=0\}$. Hence $w|_{\partial\Omega}=0$.

The case $N=3$. If $\Omega=\{|x|<1\}\subset\boldsymbol{R}^N$ and $u=u(r)$ $(r=|x|)$, the relation $-\Delta u=f(u)$ gives

(3′) $$-\Delta w=2(\omega\cdot\xi_r)f(u)-u_r\left(\Delta-2\frac{N-1}{r}\frac{\partial}{\partial r}+\frac{N-1}{r^2}\right)(\omega\cdot\xi)+f'(u)w \quad (\text{in } \Omega),$$

where $w=\nabla u\cdot\xi$ for each vector field $\xi\in C^2$ $(\Omega\to\boldsymbol{R}^N)$.

For $\Omega_+=\{(x,y,z)\in\boldsymbol{R}^3 \mid x^2+y^2+z^2<1,\ y>0\}$, we take $\xi=\xi(\rho,y)$ with $\rho=\sqrt{x^2+z^2}$, of which each section cut by a plane T containing the y-axis is similar to $[\nu]$. Obviously $\omega\cdot\xi>0$ and hence $w<0$ in Ω_+. Further, $w=0$ on $\partial\Omega_+$ is verified in a similar way. We shall show that $\omega\cdot\xi_r<0$ and

$$\left(\Delta-2\frac{N-1}{r}\frac{\partial}{\partial r}+\frac{N-1}{r^2}\right)(\omega\cdot\xi)>0 \qquad \text{in } \Omega_+.$$

Then the desired relation (2) will follow from $f(u)>0$ and $u_r<0$.

By the definition, the vector field ξ lies in each plane T containing the y-axis. Without loss of generality, suppose that T contains the x-axis, too. Then on this plane T, ξ is nothing but $[\nu]$, where $\nu(z)=i(1+z^2)$ with $z=x+iy$. Therefore, we have if $y>0$ that $\omega\cdot\xi_r=\operatorname{Re}\nu_z<0$ and

$$\left(\Delta-2\frac{N-2}{r}\frac{\partial}{\partial r}+\frac{N-1}{r}\right)(\omega\cdot\xi)=\left(\Delta_2-\frac{4}{r}\frac{\partial}{\partial r}+\frac{2}{r^2}\right)(\omega\cdot\xi)$$
$$=\left(\Delta_2-\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\right)(\omega\cdot\xi)+\left(\frac{1}{r^2}-2\frac{1}{r}\frac{\partial}{\partial r}\right)(\omega\cdot\xi)=\frac{1}{r^2}\omega\cdot\xi-\frac{2}{r}\omega\cdot\xi_r>0.$$

§ 3. An example.

We consider the nonlinear eigenvalue problem

(4) $\quad -\Delta u=\lambda e^u \text{ (in } \Omega), \qquad u=0 \text{ (on } \partial\Omega)$

in $\Omega=\{|x|<1\}\subset\boldsymbol{R}^N$, where λ is a positive parameter. Its bifurcation diagram has been completely known ([4], [6]). In particular, when $2<N<10$, the solution branch $\mathcal{S}$ in $\lambda-u$ plane starting from $(\lambda,u)=(0,0)$ bends infinitely many times around the line $\lambda=2\{(N-2)\}^{-1}$. With each point $g=(\lambda,u)\in\mathcal{S}$, the linearized operator $A(g)=-\Delta-\lambda e^u$ is associated. In the same way, the set of eigenvalues of $A(g)$ is divided as $\sigma(g)=\cup_{m=0}^{\infty}\sigma_m(g)$. From Theorem, we have $\sigma_m(g)\cap(-\infty,0]=\phi$ for $m\geqq 1$. Hence, at each bending point $\bar{g}$ in $\mathcal{S}$, the eigenvalue 0 of $A(\bar{g})$ is simple so that the local analysis of [3] works.

Thus, around $\bar{g}=(\bar{\lambda},\bar{u})$, the branch is parametrized as $g(t)=(\lambda(t), u(t))$ ($|t|$: small) with $(\lambda(0), u(0))=(\bar{\lambda},\bar{u})$, $\dot{\lambda}(0)=0$ and $\dot{u}(0)=\bar{\varphi}$, where $\bar{\varphi}\not\equiv 0$ is an eigenfunction of $A(\bar{g})$ corresponding to the simple eigenvalue 0. Therefore, we have a smooth relation in t (see [7], e.g.): $A(g(t))\varphi(t)=\mu(t)\varphi(t)$ for $|t|$: small, with $\mu(0)=0$ and $\varphi(0)=\bar{\varphi}$. From this relation, we can deduce

$$\dot{\mu}(0)\|\bar{\varphi}\|^2=\ddot{\lambda}(0)\qquad(e^{\bar{u}},\bar{\varphi}), \tag{5}$$

where $\|\cdot\|$ and $(\ ,\ )$ are the norm and inner product in $L^2(\Omega)$, respectively. Here,

$$\bar{\lambda}(e^{\bar{u}},\bar{\varphi})=\int_\Omega(-\Delta\bar{\varphi})dv=-\int_{\partial\Omega}\frac{\partial\bar{\varphi}}{\partial n}dS=-\left|\partial\Omega\right|\frac{\partial\bar{\varphi}}{\partial r}\bigg|_{r=1}\neq 0$$

so that $\dot{\mu}(0)\neq 0$ by $\ddot{\lambda}(0)\neq 0$.

In this way, we have gotten the conclusion. Along the branch $\mathcal{S}$, through each bending point the number $l=\sharp\{\sigma_0(g)\cap(-\infty,0]\}$ increases one by one and hence from 0 to infinite. This fact has been known up to the second bending ([8]).

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