88. *The Sylvester's Law of Inertia for Jordan Algebras*

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The purpose of this note is to present some results on the orbit structure of a compact (=formally real) simple Jordan algebras under the action of the identity component of its structure group. In view of the classification of compact simple Jordan algebras, Theorem 1 is viewed as a natural generalization of the Sylvester's law of inertia for real symmetric or complex Hermitian matrices. We shall use terminologies and well-known facts in the theory of Jordan algebras without giving explanations (see, for instance, Jacobson [2] and Braun-Koecher [1]).

1. Let $\mathfrak{A}$ be a compact simple Jordan algebra of degree r, and let $G(\mathfrak{A})$ be the structure group of $\mathfrak{A}$. Let $G^0(\mathfrak{A})$ denote the identity component of $G(\mathfrak{A})$. Let $a \in \mathfrak{A}$ and let

$$(1) \qquad m_a(\lambda)=\lambda^r-\sigma_1(a)\lambda^{r-1}+\cdots+(-1)^r\sigma_r(a)$$

be the generic minimum polynomial of a (for details, see [2]). Note that each $\sigma_i(a)$ is a homogeneous polynomial of degree i in the components of a. If we denote the minimum polynomial of the element a by $\mu_a(\lambda)$, then each irreducible factor of $m_a(\lambda)$ is a factor of $\mu_a(\lambda)$ ([2]). The polynomial equation $\mu_a(\lambda)=0$ has only real roots, since $\mathfrak{A}$ is compact ([1]). Therefore the equation $m_a(\lambda)=0$ also has only real roots. By the *signature* of an element $a \in \mathfrak{A}$ (denoted by sgn (a)), we mean the pair of the integers (p, q) such that p and q are numbers of positive and negative roots of the equation $m_a(\lambda)=0$, respectively. Here the number of a root should be counted by including its multiplicity. Let $\mathfrak{A}_{p,q}$ denote the set of elements $a \in \mathfrak{A}$ with sgn $(a)=(p, q)$. Then we have

$$(2) \qquad \mathfrak{A}=\coprod_{p+q\leqslant r}\mathfrak{A}_{p,q}.$$

Now let e be the unit element of $\mathfrak{A}$. Since $\mathfrak{A}$ is of degree r, one can choose a system of primitive orthogonal idempotents $\{e_1, \cdots, e_r\}$ of $\mathfrak{A}$ such that $\sum_{i=1}^r e_i=e$. Such systems are conjugate to each other under the automorphism group Aut $\mathfrak{A}$ of $\mathfrak{A}$. We choose and fix such a system $\{e_1, \cdots, e_r\}$ and put

$$(3) \qquad o_{p,q}=\sum_{i=1}^{p} e_i-\sum_{j=p+1}^{p+q} e_j, \quad p, q\geqslant 0, \quad p+q\leqslant r;$$

here we are adopting the convention that the first and the second terms of the right hand side of (3) should be zero, provided that $p=0$ and $q=0$, respectively.

Theorem 1. *Let $\mathfrak{A}$ be a compact simple Jordan algebra of degree r. Then the decomposition* (2) *is the G^0 $(\mathfrak{A})$-orbit decomposition of $\mathfrak{A}$. More* *precisely, each subset* $\mathfrak{A}_{p,q}$ *is the* $G^0(\mathfrak{A})$*-orbit through the point* $o_{p,q}$ $(p, q \geqslant 0, p+q \leqslant r)$.

Sketch of the proof. By the *rank* of an element $a \in \mathfrak{A}$ (denoted by rank (a)), we mean the number of non-zero roots of the equation $m_a(\lambda)=0$. $\mathfrak{A}_k$ denotes the set of elements $a \in \mathfrak{A}$ with rank $(a)=k$. Note that $0 \leqslant k \leqslant r$. Starting from the Jordan algebra $\mathfrak{A}$, one can construct a simple graded Lie algebra $\mathfrak{g}=\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1$, called a *symmetric Lie algebra*, with $\mathfrak{g}_{-1}$ as the underlying vector space of $\mathfrak{A}$ (Koecher [4], Kantor [3]). The adjoint representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is faithful, and further the group $G^0(\mathfrak{A})$ coincides with the analytic subgroup of $GL(\mathfrak{g}_{-1})$ corresponding to the Lie algebra $\mathrm{ad}_{\mathfrak{g}_{-1}}\mathfrak{g}_0$. By applying a result of Takeuchi [8] to the graded Lie algebra $\mathfrak{g}$, we can conclude that the set $\mathfrak{A}_k=\coprod_{p+q=k}\mathfrak{A}_{p,q}$ $(0 \leqslant k \leqslant r)$ is stable under the action of $G^0(\mathfrak{A})$. Also we use the invariance of the generic minimum polynomial $m_a(\lambda)$ under Aut $\mathfrak{A}$, and use the fact that the roots of the equation $m_a(\lambda)=0$ depend continuously on a.

Remark. (1) $\mathfrak{A}_{q,p}=-\mathfrak{A}_{p,q}$ holds.

(2) The orbit $\mathfrak{A}_{p,q}$ is open if and only if $p+q=r$. All open $G^0(A)$-orbits in $\mathfrak{A}$ have been found by Satake [7].

(3) The open $G^0(\mathfrak{A})$-orbit $\mathfrak{A}_{r,0}$ is an irreducible homogeneous self-dual convex cone, and $G^0(\mathfrak{A})$ coincides with the identity component of the automorphism group of the cone $\mathfrak{A}_{r,0}$ (Koecher [4], Vinberg [9]).

2. Since the roots of the equation $m_a(\lambda)=0$ depend continuously on a, we have the following closure relation for $G^0(\mathfrak{A})$-orbits.

Theorem 2. With assumptions in Theorem 1, let $\bar{\mathfrak{A}}_{p,q}$ denote the closure of $\mathfrak{A}_{p,q}$ in $\mathfrak{A}$. Then we have

$$\bar{\mathfrak{A}}_{p,q}=\coprod_{\substack{p_1 \leqslant p \\ q_1 \leqslant q}} \mathfrak{A}_{p_1,q_1},$$

where $p, q \geqslant 0$, $p+q \leqslant r$.

Corollary 3. Let $\partial\mathfrak{A}_{r,0}$ be the boundary of the irreducible homogeneous self-dual cone $\mathfrak{A}_{r,0}$. Then we have

$$\partial\mathfrak{A}_{r,0}=\mathfrak{A}_{r-1,0} \coprod \mathfrak{A}_{r-2,0} \coprod \cdots \coprod \mathfrak{A}_{0,0},$$

which is the stratification of $\partial\mathfrak{A}_{r,0}$ whose strata are all $G^0(\mathfrak{A})$-orbits.

3. We shall give a list of open $G^0(\mathfrak{A})$-orbits $\mathfrak{A}_{r-k,k}$ $(0 \leqslant k \leqslant r)$ in each compact simple Jordan algebra $\mathfrak{A}$. It turns out that every orbit $\mathfrak{A}_{r-k,k}$ is an affine symmetric space of K_ε-type in the sense of Oshima-Sekiguchi [6].

$\mathfrak{A}$	deg $\mathfrak{A}$	$\mathfrak{A}_{r-k,k}$ $(0 \leqslant k \leqslant r)$	
$H(r, \boldsymbol{R})(r \geqslant 3)$	r	$H^{r-k,k}(\boldsymbol{R})=GL(r, \boldsymbol{R})/O(r-k, k)$	
$H(r, \boldsymbol{C})(r \geqslant 3)$	r	$H^{r-k,k}(\boldsymbol{C})=GL(r, \boldsymbol{C})/U(r-k, k)$	
$H(r, \boldsymbol{H})(r \geqslant 3)$	r	$H^{r-k,k}(\boldsymbol{H})=GL(r, \boldsymbol{H})/Sp(r-k, k)$	
$\boldsymbol{R}^{m+2}$ $(m \geqslant 1)$	2	$C^{2-k,k}(m+2)=\boldsymbol{R}^+ \cdot O(m+1, 1)/O(m+1)$	$(k=0, 2)$
		$C^{1,1}(m+2)=\boldsymbol{R}^+ \cdot O(m+1, 1)/O(m, 1)$	$(k=1)$
$H(3, \boldsymbol{O})$	3	$H^{3-k,k}(\boldsymbol{O})=\boldsymbol{R}^+ \cdot E_{6(-26)}/F_4$	$(k=0, 3)$
		$H^{3-k,k}(\boldsymbol{O})=\boldsymbol{R}^+ \cdot E_{6(-26)}/F_{4(-20)}$	$(k=1, 2)$.

Here $H(r, \boldsymbol{F})$ denotes the compact simple Jordan algebra of Hermitian matrices of degree r with entries in the division algebra $\boldsymbol{F}=\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$ (=the quaternion algebra) or $\boldsymbol{O}$ (=the octanion algebra). $\boldsymbol{R}^{m+2}$ denotes the compact simple Jordan algebra of degree 2 of dimension $m+2$. $\boldsymbol{R}^{+}$ denotes the multiplicative group of positive real numbers.

$$H^{r-k,k}(\boldsymbol{F})=\{X \in H(r,\boldsymbol{F}) : \operatorname{sgn}(X)=(r-k,k)\},$$
$$C^{0,2}(n)=\{(x_1,\cdots,x_n)\in \boldsymbol{R}^n : x_1^2>x_2^2+\cdots+x_n^2, x_1<0\},$$
$$C^{2,0}(n)=\{(x_1,\cdots,x_n)\in \boldsymbol{R}^n : x_1^2>x_2^2+\cdots+x_n^2, x_1>0\},$$
$$C^{1,1}(n)=\{(x_1,\cdots,x_n)\in \boldsymbol{R}^n : x_1^2<x_2^2+\cdots+x_n^2\}.$$

The details of this note and its applications will be published elsewhere.

Added in proof. After this was submitted, the author found that Theorem 1 had been obtained by Satake independently (cf. I. Satake, On zeta functions associated with self-dual homogeneous cones; Reports on Symposium of Geometry and Automorphic Functions, Tohoku Univ., 145–168, 1988).

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