84. *Zeta Zeros and Dirichlet L-functions. II*

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We shall extend the investigations in [2] further. Let γ run over the positive imaginary parts of the zeros of the Riemann zeta function $\zeta(s)$. We are concerned with the distribution of $b(\gamma/2\pi)\log(\gamma/2\pi e\alpha)$ mod one. When $b>1$, the problem seems to be very difficult and our knowledge seems to be very scarce except our Theorem 5 below and a simple consequence of theorem in [1] with the help of Pjateckii-Sapiro's theorem in [4]. In this article we shall show that even the case for $0<b\leqq 1$ involves also the difficulty which lies as deep as the Generalized Riemann Hypothesis (G.R.H.) for Dirichlet L-functions $L(s,\chi)$. We assume the Riemann Hypothesis below.

We start with recalling the following fundamental theorem which is a special case of our main theorem in [1].

Theorem 1. *Let K be an integer $\geqq 1$ and let $T>T_0$. Then for any positive α,*

$$\sum_{\gamma<T} e\left(\frac{\gamma}{2\pi K}\log\left(\frac{\gamma}{2\pi e\alpha K}\right)\right) = -e^{(1/4)\pi i}\sqrt{\alpha}K \sum_{n<(T/2\pi\alpha K)^{1/K}} \Lambda(n)e(-\alpha n^K)n^{(1/2)(K-1)} + O(T^{(2/5)+(1/2K)}(\log T\cdot\log\log T)^{4/5}) + O(\sqrt{T}\log^2 T),$$

where we put $e(x)=e^{2\pi i x}$, $\Lambda(x)=\log p$ if $x=p^k$ with a prime number p and an integer $k\geqq 1$ and $\Lambda(x)=0$ otherwise.

When α is rational, we get the following corollary using the prime theorem in the arithmetic progressions.

Corollary 1. *Let K be an integer $\geqq 1$ and let $T>T_0$. Then for any integers a and $q\geqq 1$ with $(a,q)=1$, we have*

$$\sum_{\gamma<T} e\left(\frac{\gamma}{2\pi K}\log\left(\frac{\gamma}{2\pi e(a/q)K}\right)\right) = -e^{(1/4)\pi i}C\left(\frac{a}{q},K\right)(T/2\pi)^{(1/2)(1+(1/K))} + O(T^{(1/2)(1+(1/K))}\exp(-C\sqrt{\log T})),$$

where we put $C(a/q,K)=2K^{(1/2)(1-(1/K))}\overline{S(a/q,K)}(K+1)^{-1}\varphi(q)^{-1}(a/q)^{-1/2K}$ and $S(a/q,K)=\sum'^{q}_{b=1} e((a/q)b^K)$, the dash indicates that b satisfies $(b,q)=1$, C denotes some positive constant and $\varphi(q)$ is the Euler function.

When α is irrational, using the estimate due to Vinogradov of $\sum_{n<Y}\Lambda(n)e(\alpha n^K)$ (cf. [6] and also Lemma 2 in [3]), we get the following corollary to Theorem 1 and Corollary 1, which has been mentioned only for the case for $K=1$ (cf. Corollary 5 in [1]).

Corollary 2. *Let K be an integer $\geqq 1$. Then we have*

$$\lim_{T\to\infty}(T/2\pi)^{-(1/2)(1+(1/K))}\sum_{\gamma<T} e\left(\frac{\gamma}{2\pi K}\log\left(\frac{\gamma}{2\pi e\alpha K}\right)\right) = \begin{cases} -e^{(1/4)\pi i}C(a/q,K) & \text{if } \alpha=a/q \text{ with integers } a \text{ and } q\geqq 1 \text{ and } (a,q)=1 \\ 0 & \text{if } \alpha \text{ is irrational.}\end{cases}$$

It is of great interest to determine the true order of the magnitude of the remainder term in Corollary 1. In fact, we obtain the following theorem immediately from Theorem 1.

Theorem 2. *Let q be an integer $\geqq 3$. Suppose that K is an integer $\geqq 5$. Then* G.R.H. *for all $L(s,\chi^K)$ with a character χ* mod q *is equivalent to the relation*

$$\sum_{\gamma<T} e\left(\frac{\gamma}{2\pi K}\log\left(\frac{\gamma}{2\pi e(a/q)K}\right)\right) = -e^{(1/4)\pi i}C\left(\frac{a}{q}, K\right)(T/2\pi)^{(1/2)(1+(1/K))}+0(T^{1/2+\varepsilon})$$

for any positive ε and any integer a with $1\leqq a\leqq q$ and $(a,q)=1$.

Proof. Suppose first that the above relation is correct. Then using Theorem 1, we get for any character χ mod q,

$$\begin{aligned}&\sum_{n<Y} \Lambda(n)\chi^K(n)n^{(1/2)(K-1)}\\ &=\frac{1}{\overline{\tau(\chi)}}\sum_{n<Y}\Lambda(n)\left(\sum_{a=1}^{q}{}' e\left(-\frac{a}{q}n^K\right)\bar{\chi}(a)\right)n^{(1/2)(K-1)}+0(Y^{(1/2)(K-1)}\log Y)\\ &=-e^{-(1/4)\pi i}\sqrt{q}\,(K\overline{\tau(\chi)})^{-1}\sum_{a=1}^{q}{}'\frac{\bar{\chi}(a)}{\sqrt{a}}\sum_{\gamma<Y^K 2\pi(a/q)K} e\left(\frac{\gamma}{2\pi K}\log\left(\frac{\gamma}{2\pi e(a/q)K}\right)\right)\\ &\quad+0(Y^{(1/2)K}\log^2 Y)\\ &=2((K+1)\varphi(q)\overline{\tau(\chi)})^{-1}\sum_{a=1}^{q}{}'\bar{\chi}(a)\overline{S\left(\frac{a}{q}, K\right)}Y^{(1/2)K+1/2}+0(Y^{(1/2)K+\varepsilon}),\end{aligned}$$

where we put $\tau(\chi)=\sum{}'^{q}_{a=1}\chi(a)e(a/q)$.

Hence, we get

$$\begin{aligned}\sum_{n<Y}\Lambda(n)\chi^K(n)&=(\overline{\tau(\chi)}\varphi(q))^{-1}\sum_{a=1}^{q}{}'\overline{\chi(a)}\overline{S\left(\frac{a}{q}, K\right)}Y+0(Y^{1/2+\varepsilon})\\ &=\begin{cases}Y+0(Y^{1/2+\varepsilon}) & \text{if } \chi^K=\chi_0=\text{the principal character mod } q\\ 0(Y^{1/2+\varepsilon}) & \text{otherwise.}\end{cases}\end{aligned}$$

This implies G.R.H. for all $L(s,\chi^K)$. To prove the converse, we notice only that

$$\sum_{n<Y}\Lambda(n)e\left(-\frac{a}{q}n^K\right)=\frac{1}{\varphi(q)}\sum_{\chi}\chi(a)\overline{\tau(\chi)}\sum_{n<Y}\Lambda(n)\chi^K(n),$$

where χ runs over all characters mod q. Q.E.D. of Theorem 2.

We remark that this theorem should hold also for $1\leqq K\leqq 4$. To get this we have only to get rid of the term $0(T^{(2/5)+(1/2K)}(\log T\cdot\log\log T)^{4/5})$ from Theorem 1, where $(2/5)+(1/2K)\leqq(1/2)$ if $K\geqq 5$.

We turn our attentions to an infinite series involving $e(\gamma/2\pi K\cdot\log(\gamma/Ke))$. As an application of the above Corollary 1, we get immediately the following theorem.

Theorem 3. *Let K be an integer $\geqq 1$. Let a and q be integers satisfying $1\leqq a\leqq q$ and $(a,q)=1$. Then*

$$\begin{aligned}&\sum_{\gamma} e\left(\frac{\gamma}{2\pi K}\log\left(\frac{\gamma}{Ke}\right)\right)e^{-\gamma\pi/2K}\gamma^{(1/2)((1/K)-1)}\left(x+2\pi i\frac{a}{q}\right)^{-(1/K)((1/2)+i\gamma)}\\ &=-\tilde{C}\left(\frac{a}{q}, K\right)x^{-1/K}+0(x^{-1/K}\exp(-C\sqrt{\log(1/x)}))\qquad \text{as } x\longrightarrow+0,\end{aligned}$$

where we put

$$\tilde{C}\left(\frac{a}{q}, K\right)=e^{(1/4)\pi i(1-(1/K))}\Gamma\left(\frac{1}{K}\right)K^{(1/2)((1/K)-1)}(\sqrt{2\pi}\varphi(q))^{-1}\overline{S\left(\frac{a}{q}, K\right)}$$

and $\Gamma(s)$ is the Γ-function.

It is again of great interest to determine the true order of the magnitude of the remainder term of the above theorem. We can, in fact, show the following theorem.

Theorem 4. *Let K be an integer $\geqq 1$. Let q be an integer $\geqq 3$. Then* G.R.H. *for all $L(s, \chi^K)$ with a character $\chi \bmod q$ is equivalent to the relation*

$$\sum_{\gamma} e\left(\frac{\gamma}{2\pi K}\log\left(\frac{\gamma}{Ke}\right)\right)e^{-\gamma\pi/2K}\gamma^{(1/2)((1/K)-1)}\left(x+2\pi i\frac{a}{q}\right)^{-(1/K)((1/2)+i\gamma)}$$
$$=-\tilde{C}\left(\frac{a}{q}, K\right)x^{-1/K}+0(x^{-1/2K-\varepsilon}) \qquad as\ x\longrightarrow +0$$

for any positive ε and for any integer a with $1\leqq a\leqq q$ and $(a, q)=1$.

Proof. By evaluating the integral $1/2\pi i\int_{2-i\infty}^{2+i\infty}(\zeta'/\zeta)(s)\Gamma(s/K)y^{-s}ds$ in two ways and replacing y^K by $x+2\pi i(a/q)$ with a sufficiently small positive x, we get as $x\rightarrow +0$,

$$\sum_{n=2}^{\infty}\Lambda(n)e^{-n^Kx}e\left(-\frac{a}{q}n^K\right)=-\frac{1}{K}\sum_{\rho}\Gamma\left(\frac{\rho}{K}\right)\left(x+2\pi i\frac{a}{q}\right)^{-\rho/K}+0(1),$$

where ρ runs over $(1/2)+i\gamma$ and $(1/2)-i\gamma$. By the Stirling's formula, the right hand side is

$$=-e^{(1/4)\pi i((1/K)-1)}K^{-(1/2)(1+(1/K))}\sqrt{2\pi}\sum_{\gamma}e\left(\frac{\gamma}{2\pi K}\log\left(\frac{\gamma}{Ke}\right)\right)e^{-\gamma\pi/2K}\gamma^{(1/2)((1/K)-1)}$$
$$\times\left(x+2\pi i\frac{a}{q}\right)^{-(1/K)((1/2)+i\gamma)}+0\left(\sum_{\gamma}e^{-\gamma\pi/2K}\gamma^{(1/2)((1/K)-3)}\left|\left(x+2\pi i\frac{a}{q}\right)^{-(1/K)((1/2)+i\gamma)}\right|\right).$$

The last term is seen to be $0(x^{-\varepsilon})$ as $x\rightarrow +0$. Suppose first that G.R.H. holds for all $L(s, \chi^K)$ with a character $\chi \bmod q$. Then for any $(a, q)=1$,

$$\sum_{n=2}^{\infty}\Lambda(n)e^{-n^Kx}e\left(-\frac{a}{q}n^K\right)$$
$$=\frac{1}{\varphi(q)}\sum_{\chi}\overline{\tau(\chi)}\chi(a)\int_1^{\infty}e^{-y^Kx}d\left(\sum_{n<y}\Lambda(n)\chi^K(n)\right)$$
$$=\frac{1}{\varphi(q)}\sum_{\chi}{}''\,\overline{\tau(\chi)}\chi(a)\int_1^{\infty}e^{-y^Kx}dy+0(x^{-(1/2K)-\varepsilon})$$
$$=\frac{1}{\varphi(q)}\sum_{\chi}{}''\,\overline{\tau(\chi)}\chi(a)\Gamma\left(\frac{1}{K}\right)K^{-1}x^{-1/K}+0(x^{-(1/2K)-\varepsilon}),$$

where the double dash indicates that χ satisfies $\chi^K=\chi_0$. Conversely, assume the last asymptotic formula for any $(a, q)=1$. Then for any character $\chi \bmod q$,

$$\frac{L'}{L}(Ks, \chi^K)=-(\Gamma(s)\overline{\tau(\chi)})^{-1}\sum_{a=1}^{q}{}'\,\bar{\chi}(a)\int_0^{\infty}\left(\sum_{n=2}^{\infty}\Lambda(n)e^{-n^Kx}e\left(-\frac{a}{q}n^K\right)\right)x^{s-1}dx$$
$$=-\Gamma\left(\frac{1}{K}\right)(K\Gamma(s)\overline{\tau(\chi)}\varphi(q))^{-1}\sum_{\chi'}{}''\,\overline{\tau(\chi')}\left(\sum_{a=1}^{q}{}'\,\chi'(a)\bar{\chi}(a)\right)\int_0^{\eta}x^{s-1-(1/K)}dx$$
$$+G(s),$$

where η is a sufficiently small positive number and $G(s)$ is regular for $\mathrm{Re}(s)>1/2K$. Thus we see that $(L'/L)(Ks, \chi^K)$ is regular for $\mathrm{Re}(s)>1/2K$ except when $Ks=1$ and $\chi^K=\chi_0$. Since $\sum_\chi'' \overline{\tau(\chi)}\chi(a)=\overline{S(a/q,K)}$, by Lemma 1 of [2], we get our Theorem 4. Q.E.D.

This gives us a generalization of Sprindzuk's theorem, namely, for the case when $K=1$, in [5]. On the other hand, we may recall that we have extended Sprindzuk's theorem in another direction as follows, where we shall correct the statement of Theorem 2 for $K\geqq 3$ in [2] on this occasion.

Theorem 5. *Let q be an integer $\geqq 3$. Let K be an integer $\geqq 2$. Then* G.R.H. *for all $L(s,\chi)$ with a character χ mod q is equivalent to the relation*

$$\sum_\gamma e\left(\frac{K\gamma}{2\pi}\log\left(\frac{K\gamma}{e}\right)\right)e^{-(1/2)\pi\gamma K}\gamma^{(1/2)(K-1)}\left(x+2\pi i\frac{a}{q}\right)^{-K((1/2)+i\gamma)}\left(1+\frac{A_1}{\gamma}+\cdots+\frac{A_{K_0}}{\gamma^{K_0}}\right)$$

$$+B(K)\sum_{d=1,\, d\neq K}^{2k-1}\sum_p \log p\cdot e^{-xp^{d/K}}e\left(-\frac{a}{q}p^{d/K}\right)$$

$$=-\frac{1}{x}B(K)\frac{\mu(q)}{\varphi(q)}+0(x^{-(1/2)-\varepsilon})$$

as $x\to+0$ for any positive ε and for any integer a with $1\leqq a\leqq q$ and $(a,q)=1$, where $B(K)=(2\pi)^{-1/2}K^{-(1/2)(K+1)}e^{-(1/4)\pi i(K-1)}$, $K_0=[(1/2)(K-1)]$ if $K\geqq 3$, $A_1, \cdots, A_{K_0}$ are the constants which may depend on K, $A_1=\cdots=A_{K_0}=0$ if $K=2$, p runs over the primes and $\mu(q)$ is the Möbius function.

We remark that $A_1, \cdots$ and A_{K_0} can be written down explicitly.

References

[1] A. Fujii: On the uniformity of the distribution of the zeros of the Riemann zeta function. II. Comment. Math. Univ. St. Pauli, **31** (1), 99–113 (1982).

[2] ——: Zeta zeros and Dirichlet L-functions. Proc. Japan Acad., **64**A, 215–218 (1988).

[3] ——: Some additive problems of numbers. Banach Center Publ., **17**, 121–141 (1985).

[4] I. I. Pjateckii-Sapiro: On the distribution of prime numbers in sequences of the form $[f(n)]$. Math. Sb., **33**, 559–566 (1953).

[5] V. G. Sprindzuk: Vertical distribution of the zeros of zeta function and extended Riemann hypothesis. Acta Arith., XXVII, pp. 317–332 (1975).

[6] I. M. Vinogradov: Selected Works. Springer (1985).