82. On Generalization of G_δ Diagonal and Metrization

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Introduction. J. Chaber's result, [6], that countably compact spaces with G_δ diagonal are compact, has inspired us to define the conditions (α_i), $i=1,2$ as follows. (See J.G. Ceder's characterization of G_δ diagonal, [5].)

(α_1) (resp. (α_2)): There is a sequence $\{\mathcal{U}_n\}_n$ of semi-open covers with $\bigcap_n St(x, \mathcal{U}_n)=\{x\}$, (resp. $\bigcap_n \overline{St(x, \mathcal{U}_n)}=\{x\}$) for each x in the topological space under consideration, where $St(x, \mathcal{U}_n)=\bigcup\{U\in\mathcal{U}_n \mid x\in U\}$. We call such a sequence $\{\mathcal{U}_n\}_n$ an (α_1) (resp. (α_2)) sequence.

According to [1], a cover $\mathcal{U}$ of a space x is called *semi-open*, if $x\in \operatorname{Int} St(x, \mathcal{U})$ for $x\in X$. Clearly (α_2) and G_δ diagonal property [5] imply (α_1).

1. Implication of the conditions (α_i), $i=1,2$. Refer [2], [11] and [12] respectively for the notions of point countable type, q and wM.

Lemma 1.1. *A regular q space X with (α_1) is first countable.*

Proof. Let $x\in X$, $\{G_n\}_n$ be a q sequence at x and $\{\mathcal{U}_n\}_n$ be an (α_1) sequence in X. By induction, we can obtain a sequence $\{H_n\}_n$ of open sets with $x\in H_{n+1}\subset \overline{H}_{n+1}\subset H_n\cap G_{n+1}\cap \operatorname{Int} St(x, \mathcal{U}_{n+1})$ for each n. Now, $\{H_n \mid \geqslant 1\}$ forms a local base at x. Q.E.D.

Theorem 1.2. *A regular q space with (α_1) is an open continuous image of a metrizable space.*

Proof. Apply the Lemma 1.1 and a result of Ponomarev and Hanai [13], [8]. Q.E.D.

Theorem 1.3. *If a regular q space with (α_1) is a quotient image of a locally compact, separable and metrizable space, then the space is locally compact, separable and metrizable.*

Proof. Apply the Lemma 1.1 and a result of A. H. Stone, [15]. Q.E.D.

Note 1.4. Since a T_1 space of point countable type is a q space [9], we can replace "q space" by "space of point countable type" in the above theorem.

Theorem 1.5. *A wM space X is metrizable, iff it has (α_2).*

Proof. The condition is clearly necessary. To prove that the condition is sufficient, let X have (α_2) and $\{\mathcal{U}_n\}_n$ be an (α_2) sequence. Let $\{\mathcal{G}_n\}_n$ be a wM sequence in X. Let $\mathcal{H}_n=\mathcal{U}_n\wedge\mathcal{G}_n$ for each $n\geqslant 1$, where $\mathcal{U}_n\wedge\mathcal{G}_n=\{U\cap G \mid U\in\mathcal{U}_n \text{ and } G\in\mathcal{G}_n\}$. Then $\{\mathcal{H}_n\}_n$ is also an (α_2) sequence. We may assume that $\mathcal{H}_{n+1}\prec\mathcal{H}_n$ (i.e. $\mathcal{H}_{n+1}$ refines $\mathcal{H}_n$) for each n. To claim that $\{\mathcal{H}_n\}_n$ is a semi-development of X, we show that $\{St(x, \mathcal{H}_n) \mid n\geqslant 1\}$ is a network at x

for each $x \in X$. Let $x \in X$ and U be an open nhd of x. Assume that $x_n \in St(x, \mathcal{H}_n) - U$ for each n. Then $\{x_n\}_n$ has a cluster point, say, y. Since $\mathcal{H}_{n+1} \prec \mathcal{H}_n$ for each n, we have $y \in \overline{St(x, \mathcal{H}_n)}$ for each n. It follows that $y \in \bigcap_n \overline{St(x, \mathcal{H}_n)} = \{x\} \subset U$. Since U is an open nhd of y and y is a cluster point of $\{x_n\}_n$, it follows that U contains infinitely many x_n's, which is a contradiction to $x_n \notin U$ for each n. Therefore $St(x, \mathcal{H}_n) \subset U$ for some n. Thus X is a T_2 semi-developable space and hence it is semi-metrizable, [1]. As a T_2 semi-metrizable wM space is metrizable, X is metrizable, [11]. Q.E.D.

Note 1.6. A $w\Delta$ space with (α_2) is semi-metrizable and a regular $w\Delta$ space with (α_2) is a Moore space. (See [10] for $w\Delta$ notion.) An (α_1) sequence $\{\mathcal{U}_n\}_n$ is called an (α_3) sequence if each $\mathcal{U}_n$ is a closure preserving closed cover. A $\sum^{\#}$ space with (α_3) is a σ-space.

2. Invariance of Semi-developability. According to [3], a map $f: X \to Y$ is called a *pseudo-open map*, if for each $y \in Y$, whenever U is a nhd of $f^{-1}(y)$, $f(U)$ is a nhd of y. It is well-known that G_δ diagonal property is not preserved under perpect maps, [4] (even more, under finite to one closed continuous maps [14]).

Lemma 2.1. *Let $f: X \to Y$ be a finite to one, pseudo-open map and let X have (α_1). Then Y has (α_1).*

Proof. Let $\{\mathcal{U}_n\}_n$ be an (α_1) sequence in X, and $\mathcal{U}_{n+1} \prec \mathcal{U}_n$ for each $n \geqslant 1$. Let $\mathcal{G}_n = f(\mathcal{U}_n) = \{f(U) \mid U \in \mathcal{U}_n\}$ for each $n \geqslant 1$. We claim that $\{\mathcal{G}_n\}_n$ is an (α_1) sequence in Y. To see $\mathcal{G}_n$ is a semi-open cover of Y for each n, let $y \in Y$. Since, $St(y, \mathcal{G}_n) = f(St(f^{-1}(y), \mathcal{U}_n))$ and $St(f^{-1}(y), \mathcal{U}_n)$ is a nhd of $f^{-1}(y)$, it follows that $St(y, \mathcal{G}_n)$ is a nhd of y. To see that $\bigcap_n St(y, \mathcal{G}_n) = \{y\}$ for each $y \in Y$, assume the contrary. Then, for some $y \in Y$, there is $z \in Y$ with $y \neq z$ and $z \in \bigcap_n St(y, \mathcal{G}_n)$. Consequently, for each n, we can find $U_n \in \mathcal{U}_n$ with $y, z \in f(U_n)$. Since, $f^{-1}(y) \cup f^{-1}(z)$ is finite, there is a pair of distinct points, $\{x_1, x_2\}$ contained in $U_{n_k} \in \mathcal{U}_{n_k}$ for each k, for some subsequence $\{n_k\}_k$. Since, $\mathcal{U}_{n+1} \prec \mathcal{U}_n$ for each n, we have $\{x_1, x_2\} \subset$ some $U \in \mathcal{U}_n$ for each n. But then, $x_1 = x_2$ which is a contradiction. Therefore, $\bigcap_n St(y, \mathcal{G}_n) = \{y\}$ for each $y \in Y$ and our claim is established. Q.E.D.

Corollary 2.2. *Let X have G_δ diagonal and let $f: X \to Y$ be a finite to one, pseudo-open map of X onto Y. Then Y has (α_1).*

Theorem 2.3. *Let $f: X \to Y$ be a continuous, pseudo-open, finite to one map of a T_1 semi-developable space X onto Y. Then, Y is a T_1 semi-developable space.*

Proof. Let $\{\mathcal{G}_n\}_n$ be a semi-development of X. Since X is T_1, we have $\bigcap_n St(x, \mathcal{G}_n) = \{x\}$ for each $x \in X$. From the Lemma 2.1 $\{f(\mathcal{G}_n)\}_n$ is an (α_1) sequence in Y and hence Y is T_1. We may assume that $\mathcal{G}_{n+1} \prec \mathcal{G}_n$ for each n. To claim that $\{f(\mathcal{G}_n)\}_n$ is a semi-development of Y, let $y \in Y$, and U be an open nhd of y. Since $f^{-1}(y) \subset f^{-1}(U)$, $f^{-1}(y)$ is finite, $f^{-1}(U)$ is open and $\mathcal{G}_{n+1} \prec \mathcal{G}_n$ for each n, it follows that $St(f^{-1}(y), \mathcal{G}_n) \subset f^{-1}(U)$ for some n, and consequently $St(y, f(\mathcal{G}_n)) \subset U$. Therefore $\{f(\mathcal{G}_n)\}_n$ is a semi-development in Y. Q.E.D.

Corollary 2.4. *Let $f: X \to Y$ be a continuous, finite to one, pseudo-open map of a Moore space X onto Y. Then Y is a T_1 semi-developable space.*

Note 2.5. Since closed maps are pseudo-open, (α_1) condition and T_1 semi-developability are preserved under closed, continuous, finite to one maps.

3. Examples. **Example 3.1.** There is a hereditarily paracompact space with (α_1) but it does not have G_δ diagonal. The image of the Michael line under the Slaughter's map, [14], is such a space. This space has (α_1), by the Corollary 2.2, but it does not have G_δ diagonal, [14].

Example 3.2. G_δ diagonal, quasi-G_δ diagonal and (α_1) properties are not preserved under one to one, continuous compact maps. Let N denote the space of non-zero natural numbers with the topology generated by $\{\{2n-1, 2n\} \mid n \geqslant 1\}$. Then, N is a non-T_1 second countable, developable space. Let $D(\aleph_0) = \{x_n \mid n \geqslant 1\}$ denote a discrete space of power $\aleph_0$ and let $g: D(\aleph_0) \to N$ be defined by $g(x_n) = n$ for each n. Then g is a one to one, continuous compact map of $D(\aleph_0)$ onto N. Since N is not T_1, it follows that it does not have (α_1), and quasi-G_δ diagonal.

Example 3.3. A T_2 compact space satisfying (α_1) need not be metrizable. Let $A(I) = I \cup I'$ be the Alexandroff duplicate of the unit closed interval $I = [0, 1]$. See [7] for the description of $A(I)$. $A(I)$ is a non-metrizable, T_2 compact space. Let $U_n(x) = (x - 1/n, x + 1/n) \cap I$ for each $x \in I$ and $n \geqslant 1$. $I' = \{x' \mid x \in I\}$. Let $G_n(x) = \{\{y', x\} \mid y \in U_n(x) - \{x\}\}$, $\mathcal{U}_n(I) = \{U_n(x) \mid x \in I\}$ and $\mathcal{U}_n(I') = \cup \{G_n(x) \mid x \in I\}$ for each $n \geqslant 1$. Let $\mathcal{U}_n = \mathcal{U}_n(I) \cup \mathcal{U}_n(I')$ for each $n \geqslant 1$. Then, for each $x \in I$, $St(x, \mathcal{U}_n) = [(x - 2/n, x + 2/n) \cap I] \cup \{y' \mid y \in U_n(x) - \{x\}\}$, $St(x', \mathcal{U}_n) = \cup \{\{x', y\} \mid y \in U_n(x) - \{x\}\}$. We can verify that $\{\mathcal{U}_n\}_n$ is an (α_1) sequence in $A(I)$.

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