81. On Complexes in a Finite Abelian Group. II

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This is continued from [0].

Proof of Theorem 2. Let $a \in K$ and put $K' = K - a$. Then $K+K = K \circ K$ means $K'+K' = K' \circ K'$, $|K'| = k$, $|K' \circ K'| = m$. We have $0 \in K'$, and "$K+K$ is a coset of a subgroup of G" means "$K'+K'$ is a subgroup of G". So rewriting K for K', Theorem 2 can be reformulated as follows.

Theorem 2′. *Let $0 \in K$ and suppose $K+K = K \circ K$. If $2m < 3k$, then $K+K$ is a subgroup of G.*

The proof of this theorem depends on the following

Theorem of Kneser ([1], see Mann [2, p. 6]). *For any complexes A, B of G, there exists a subgroup H of G such that $A+B = A+B+H$ and $|A+B| \geq |A+H| + |B+H| - |H|$.*

For $A = B = K$, we obtain a subgroup H such that $K+K = K+K+H$ and $|K+K| \geq 2|K+H| - |H|$. If $2m < 3k$, we have $m = |K+K| < (3/2)k \leq (3/2)|K+H|$, and so $2|H| > |K+H|$. As $K+K = K+K+H$, we have $K+K \supset H$. If $(K+K) \setminus H \neq \emptyset$, there should be $x, y \in K$ such that $x+y \notin H$. Then x or $y \notin H$. Suppose $x \notin H$. Then $K+H \supset (x+H) \cup H$ and $|K+H| \geq 2|H|$. Thus $K+K = H$.

Remark. If $G = Z/pZ$, p being a prime, then $K+K = G$ or $|K+K| \geq 2|K| - 1$. This follows from the theorem of Kneser or from Cauchy-Davenport's theorem (see Mann [2, p. 3]).

Let G be any other abelian group than $Z|pZ$ and H a non-trivial subgroup of G (i.e. $H \neq \{0\}$, $H \neq G$). Put $K = H \cup (x+H)$, $2x \notin H$. Then $|K+K| = (3/2)|K|$, so that 3/2 in (ii) can not be replaced by a smaller number.

Since $K \circ K \neq K+K$, in order to prove Theorem 3 we may suppose K satisfies (0). Moreover we may prove Theorem 3 for K with the following maximality property: there is no $s \in G \setminus K$ such that

$$(*) \qquad |(K \cup \{s\}) \circ (K \cup \{s\})| \leq |K \circ K| + 1.$$

In fact, if there exists $s \in G \setminus K$ which satisfies $(*)$, then $K' = K \cup \{s\}$ satisfies (0) and if Theorem 3 is proved with respect to K', then the inequality also holds true for K.

Lemma 4. *If $|G|$ is odd, K satisfies* (0) *and there is no $s \in G \setminus K$ which satisfies $(*)$, then $|K^w| \leq m-k+3$ for every $w \in (K \circ K) \setminus K$.*

Proof. Suppose $|K^w| \geq m-k+4$ for some $w \in (K \circ K) \setminus K$. Put $K_{(x)} = \{y \in K \setminus \{x, 0\} \mid x+y \in K\}$ for $x\ (\neq 0) \in K$, then $K_{(x)} \to K \cap K_x$, $y \to x+y$ is a

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bijection. Thus $|K^w|+|K_{(x)}|=|K^w|+|K\cap K_x|\geq m-k+4+2k-3-m=k+1$, in virtue of what we noticed just before Lemma 2. Therefore $|K^w\cap K_{(x)}| =|K^w|+|K_{(x)}|-|K^w\cup K_{(x)}|\geq k+1-(k-1)=2$.

Let $y_1, y_2\in K^w\cap K_{(x)}$, $(y_1\neq y_2)$, then $w=y_1+z_1=y_2+z_2$, $z_1\in K\setminus\{y_1, 0\}$, $z_2 \in K\setminus\{y_2, 0\}$, $z_1\neq z_2$, $x+y_1$, $x+y_2\in K$, y_1, $y_2\in K\setminus\{x, 0\}$, $w+x=(x+y_1)+z_1= (x+y_2)+z_2$. If $x+y_1=z_1$ and $x+y_2=z_2$, then $z_1+z_1=z_2+z_2$, which is a contradiction, since $|G|$ is odd. Hence $x+y_1\neq z_1$ or $x+y_2\neq z_2$, that is, $w+x\in K\circ K$ for every x $(\neq 0)\in K$. Therefore $(K\cup\{w\})\circ(K\cup\{w\})=K\circ K$ contradicting with the maximality property.

Proof of Theorem 3. We may suppose the maximality property. As $|G|$ is odd, K contains no involution. Thus the argument in the proof of Lemma 3 is valid and using Lemma 4 we obtain in the same way

$$(k-1)(k-2)\leq(k-1)(m-k+1)+(m-k+1)(m-k+3)$$

where, as $m\geq 1$,

$$m\geq\frac{1}{2}(k-3+\sqrt{5k^2-10k+9})=\frac{\sqrt{5}+1}{2}k-\frac{\sqrt{5}+3}{2}+O\left(\frac{1}{k}\right).$$

Remark. Here also we have $m\geq(3/2)k$ for $k\geq 22$.

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