62. *Zeta Zeros and Dirichlet L-functions*

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§ 1. **Introduction.** Let γ run over the imaginary parts of the non-trivial zero of the Riemann zeta function $\zeta(s)$. We have conjectured that $(b\gamma/2\pi)\log(\gamma/2\pi e\alpha)$ is uniformly distributed mod one for any positive b and α when γ runs over the positive parts. Although this is far beyond our present knowledge, we might be in a position to understand satisfactorily the exponential sum of $(b\gamma/2\pi)\log(|\gamma|/2\pi e\alpha)$ for any positive $b\leqq 1$ and any positive α by the theorem of the author's [1]. The purpose of the present article is to show that the distribution of $(b\gamma/2\pi)\log(|\gamma|/2\pi e\alpha)$ mod 1 causes much complications unless $b=1$. Moreover this will be realized in connection with the Dirichlet L-functions $L(s, \chi)$. We assume the Riemann Hypothesis throughout this article.

We recall two theorems for the case $b=1$. First, in the previous article [2] we have shown, in the corrected form, the following theorem as a consequence of the author's result which states that

$$\lim_{T\to\infty}\frac{1}{T}\sum_{0<\gamma\leqq T} e\left(\frac{\gamma}{2\pi}\log\frac{\gamma}{2\pi e\alpha}\right)=\begin{cases}-e^{(\pi/4)i}\dfrac{C(\alpha)}{2\pi} & \text{if } \alpha \text{ is rational}\\ 0 & \text{if } \alpha \text{ is irrational,}\end{cases}$$

where we put $e(x)=e^{2\pi i x}$ and $C(\alpha)=\mu(q)/\varphi(q)\sqrt{\alpha}$ with the Möbius function $\mu(q)$ and the Euler function $\varphi(q)$ when $\alpha=a/q$ with relatively prime integers a and $q\geqq 1$ (cf. [1]).

Theorem A. *Let $L(s, \chi)$ be a Dirichlet L-function with a primitive character χ mod $q\geqq 3$. Then we have*

$$\lim_{T\to\infty}\frac{2\pi}{T}\sum_{0<\gamma\leqq T}\left(L\left(\frac{1}{2}+i\gamma,\ \chi\right)-1\right)=-L(1,\bar{\chi})\chi(-1)\tau(\chi)\frac{\mu(q)}{\varphi(q)}+\frac{L'}{L}(1,\chi),$$

where we put $\tau(\chi)=\sum_{a=1}^{q}\chi(a)e(a/q)$.

Second, Sprindzuk [5] has shown the following theorem by extending Linnik's works [4].

Theorem B. *Let q be an integer $\geqq 3$. The Generalized Riemann Hypothesis* (G.R.H.) *for all $L(s, \chi)$ with a character χ mod q is equivalent to the relation*

$$\sum_{\gamma} e\left(\frac{\gamma}{2\pi}\log\frac{|\gamma|}{e}\right)e^{-(1/2)\pi|\gamma|}\left(x+2\pi i\frac{a}{q}\right)^{-(1/2)-i\gamma}=-\frac{\mu(q)}{\sqrt{2\pi}\varphi(q)}\frac{1}{x}+0(x^{-(1/2)-\varepsilon})$$

as $x\to+0$ for any positive ε and any integer a with $0<|a|\leqq q/2$, $(a, q)=1$.

We shall extend Theorems A and B in the direction of $b<1$ and $b>1$, respectively.

Theorem 1. *Let $L(s,\chi)$ be a Dirichlet L-function with a primitive* *character* χ mod $q\geqq 3$. *Let K be an integer* $\geqq 2$. *Then*

$$\lim_{T\to\infty}\left(\frac{2\pi}{Tq}\right)^{(3/4)+(1/4K)}\left(\sum_{0<\gamma\leqq T}\left(L\left(\frac{1}{2}+i\frac{\gamma}{K},\chi\right)-1\right)-\frac{T}{2\pi}\frac{L'}{L}\left(\frac{K+1}{2},\chi\right)\right)$$
$$=\begin{cases}C(K,q)\ (\neq 0) & \text{if } \chi^K=\chi_0\\ 0 & \text{otherwise,}\end{cases}$$

where we put $C(K,q)=4(K+1-2\sqrt{q}K^{1/2-(1/2K)})/q(3K+1)(K-1)$ *and* χ_0 *is the principal character* mod q.

This is a consequence of our result on the exponential sum

$$\sum_{0<\gamma\leqq T} e\left(\frac{1}{2\pi K}\gamma\log\frac{\gamma}{2\pi e\alpha}\right)\qquad \text{for } K\geqq 2.$$

The corresponding theorem for $\zeta(s)$ is also interesting and the result may be stated as follows.

Theorem 1′. *Let K be an integer* $\geqq 2$. *Then*

$$\lim_{T\to\infty}\frac{2\pi}{T}\sum_{0<\gamma\leqq T}\left(\zeta\left(\frac{1}{2}+i\frac{\gamma}{K}\right)-1\right)=\frac{\zeta'}{\zeta}\left(\frac{K+1}{2}\right).$$

We next state our extension of Sprindzuk's Theorem B.

Theorem 2. *Let q be an integer* $\geqq 3$. *Let K be an integer* $\geqq 2$. *Then* G.R.H. *for all* $L(s,\chi)$ *with a character* χ mod q *is equivalent to the relation*

$$\sum_{\gamma>0} e\left(\frac{K\gamma}{2\pi}\log\frac{K\gamma}{e}\right)e^{-(1/2)\pi\gamma K}\gamma^{(K-1)/2}\left(x+2\pi i\frac{a}{q}\right)^{-(1/2)K-Ki\gamma}$$
$$+B(K)\sum_{d=1,d\neq K}^{2K-1}\sum_{p}\log p\cdot e^{-xp^{d/K}}e\left(-\frac{a}{q}p^{d/K}\right)$$
$$=-\frac{1}{x}B(K)\frac{\mu(q)}{\varphi(q)}+0(x^{-(1/2)-\varepsilon})$$

as $x\to+0$, *for any positive* ε *and for any integer a with* $0<a\leqq q$, $(a,q)=1$, *where we put* $B(K)=(2\pi)^{-1/2}K^{-(K+1)/2}e^{-(\pi/4)i(K-1)}$ *and p runs over the primes.*

We denote the first and the second term in the left hand side of the last relation by $\mathfrak{S}_3$ and $\mathfrak{S}_{\mathfrak{P}}$ respectively. It might be that neither $\mathfrak{S}_3$ nor $\mathfrak{S}_{\mathfrak{P}}$ is $0(x^{-(1/2)-\varepsilon})$, because it is expected that

$$\sum_{0<\gamma\leqq T} e\left(\frac{K\gamma}{2\pi}\log\frac{K\gamma}{2\pi ea/q}\right)\sim CT^{\theta}$$

with $0<\theta\leqq 1$, in which case we should have $\mathfrak{S}_3\sim C'x^{-(\theta-1+(K+1)/2)}$ and $\mathfrak{S}_{\mathfrak{P}}$ must give the same order of the magnitude as $x\to+0$ and cancel each other up to $-(1/x)B(K)\mu(q)/\varphi(q)$.

§2. **Proof of Theorem 1.** Let χ be a primitive character mod $q\geqq 3$. Using the approximate functional equation of $L(s,\chi)$ (cf. p. 93 of Lavrik [3]) and a modified version of Theorems 1′ and 2′ in [2] whose proof is implicit in the author's [1], we get for $k=1/K$, $\delta=1/\log T$ and $T>T_0$,

$$\sum_{0<\gamma\leqq T}\left(L\left(\frac{1}{2}+ik\gamma,\chi\right)-1\right)$$
$$=-\frac{1}{2\pi}\sum_{n\leqq\sqrt{qTk/2\pi}}\chi(n)n^{((1/2)+\delta)k-(1/2)}k\log n\sum_{m=2,m^K=n}^{\infty}\frac{\Lambda(m)}{m^{1+\delta}\log m}\left(T-\frac{2\pi n^2K}{q}\right)$$

$$-B(\chi)\sum_{n\leqq\sqrt{qTk/2\pi}}\bar{\chi}(n)\sum_{n^k<l<(qkT/2\pi n)^k}\Lambda(l)e\left(\frac{nl^K}{q}\right)l^{(K-1)/2}+0(T^{(3+k)/4}e^{-C\sqrt{\log T}})$$

$$=S_1-B(\chi)S_2+0(T^{(3+k)/4}e^{-C\sqrt{\log T}}),$$

say, where C denotes a positive absolute constant and we put

$$B(\chi)=\sqrt{K/q}\,\tau(\chi)\chi(-1)/2q$$

and $\Lambda(x)$ is the von-Mangoldt function.

It is easily seen that

$$S_1=\frac{T}{2\pi}\frac{L'}{L}\left(\frac{K+1}{2},\chi^K\right)+\delta(\chi^K,\chi_0)(T/2\pi)^{(3+k)/4}(2K(qk)^{(3+k)/4}/q(3K+1)$$

$$+2(qk)^{(-1+k)/4}/(K-1))+0(T^{(3+k)/4}e^{-C\sqrt{\log T}}),$$

where $\delta(\chi^K,\chi_0)=1$ if $\chi^K=\chi_0$ and $=0$ otherwise.

$$\frac{1}{2}\varphi(q)(K+1)S_2=(qkT/2\pi)^{(k+1)/2}\sum_{n\leqq\sqrt{qTk/2\pi}}\frac{\bar{\chi}(n)}{n^{(1+k)/2}}\sum_{\substack{b=1\\(b,q)=1}}^{q}e\left(\frac{nb^K}{q}\right)$$

$$\sum_{n\leqq\sqrt{qTk/2\pi}}\bar{\chi}(n)n^{(1+k)/2}\sum_{\substack{b=1\\(b,q)=1}}^{q}e\left(\frac{nb^K}{q}\right)+0(T^{(1/4)(3+k)}e^{-C\sqrt{\log T}}).$$

The last inner sum is evaluated by the following lemma.

Lemma 1. *Let q be an integer $\geqq 2$ and K be an integer $\geqq 1$ and suppose that $(n,q)=1$. Then*

$$\sum_{b=1,(b,q)=1}^{q}e\left(\frac{nb^K}{q}\right)=\sum_{\chi'}\chi'(n)\tau(\bar{\chi}'),$$

where χ' runs over all characters mod q *for which $\chi'^K=\chi_0$.*

This is an immediate consequence of the following lemma which can be easily proved.

Lemma 2. *Let q be an integer $\geqq 2$ and K be an integer $\geqq 1$. Then for any c, $(c,q)=1$, we have*

$$\sum_{(b,q)=1,b^K\equiv c(q)}\cdot 1=\sum_{\chi'}\chi'(c),$$

where χ' runs over all characters mod q *for which $\chi'^K=\chi_0$.*

Using Lemma 1, we get

$$S_2=\delta(\chi^K,\chi_0)\tau(\bar{\chi})\left(\frac{qkT}{2\pi}\right)^{(3+k)/4}\cdot 8K/(q(K-1)(3K+1))+0((qT)^{(1+k)/2}).$$

Combining these results, we get Theorem 1 with the remainder term.

§3. Proof of Theorem 2. Let $\Gamma(s)$ be the Γ-function. By evaluating the integral

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{\zeta'}{\zeta}(s)\Gamma(ks)y^{-s}ds$$

in two ways, we get the following explicit formula for any $y>0$ and any integer $K\geqq 1$,

$$\sum_{n=2}^{\infty}\Lambda(n)e^{-(yn)^{1/K}}=-K\sum_{\rho}\Gamma(K\rho)y^{-\rho}+\Phi(y),\qquad \text{where } \rho$$

runs over $(1/2)+i\gamma$ and we put

$$\Phi(y)=\frac{K!}{y}+\sum_{n=1}^{\infty}\frac{y^{2n}}{(2nK)!}\left(\log y+K\frac{\Gamma'}{\Gamma}(1+2nK)-\log\pi\right.$$

$$\left.+\frac{\zeta'}{\zeta}(1+2n)+\frac{1}{2}\frac{\Gamma'}{\Gamma}(1+n)+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+n\right)\right)+\sum_{m=0,K\nmid m}^{\infty}\frac{(-1)^m}{m!}\frac{\zeta'}{\zeta}\left(-\frac{m}{K}\right)y^{m/K}.$$

We may replace $y^{1/K}$ by $x+iv$, where $v=2\pi a/q$, $(a, q)=1$, $a, q\geqq 1$ and x is a sufficiently small positive number. We see easily that

$$\sum_{n=2}^{\infty}\Lambda(n)e^{-xn^{1/K}}e\left(-\frac{a}{q}n^{1/K}\right)=\sum_{p}\log p\cdot e^{-xp}\cdot e\left(-\frac{a}{q}p\right)+\mathfrak{S}_{\mathfrak{P}}+O(x^{-1/2}\log(1/x)).$$

Since $\Phi((x+iv)^K)=O(1)$, we get

$$\sum_{p}\log p\cdot e^{-xp}\cdot e\left(-\frac{a}{q}p\right)=-\mathfrak{S}_{\mathfrak{P}}-K\sum_{\rho}\Gamma(K\rho)(x+iv)^{-K\rho}+O(x^{-(1/2)-\varepsilon})$$
$$=-F(x;K,a/q)+O(x^{-(1/2)-\varepsilon}),\quad \text{say.}$$

Suppose first that $F(x;K,a/q)=-(\mu(q)/\varphi(q))(1/x)+O(x^{-(1/2)-\varepsilon})$ for all a satisfying $(a, q)=1$ and $1\leqq a\leqq q$. Then for $\operatorname{Re}s>1$ and for a Dirichlet character χ mod q,

$$\frac{L'}{L}(s,\chi)=-\frac{1}{\Gamma(s)\overline{\tau(\chi)}}\sum_{a=1}^{q}\bar{\chi}(a)\int_0^{\infty}\left(\sum_{p}\log p\cdot e^{-xp}e\left(-\frac{a}{q}p\right)\right)x^{s-1}dx+R_1(s)$$
$$=-\frac{1}{\Gamma(s)\overline{\tau(\chi)}}\sum_{a=1}^{q}\bar{\chi}(a)\int_0^{\eta}\left(\frac{\mu(q)}{\varphi(q)}\frac{1}{x}+O(x^{-(1/2)-\varepsilon})\right)x^{s-1}dx+R_2(s),$$

where $R_1(s)$ and $R_2(s)$ are regular for $\operatorname{Re}s>(1/2)$ and η is a small positive number. The last expression represents a regular function in $\operatorname{Re}s>(1/2)$ unless $\chi=\chi_0$ and $s=1$.

Conversely, if G.R.H. holds for all $L(s,\chi)$ with a character χ mod q, then for any $(a, q)=1$,

$$-F(x;K,a/q)=\frac{1}{\varphi(q)}\sum_{b=1,(b,q)=1}e\left(-\frac{a}{q}b\right)\sum_{\chi\neq\chi_0}\bar{\chi}(b)\sum_{n=2}^{\infty}\chi(n)\Lambda(n)e^{-xn}$$
$$+\frac{\mu(q)}{\varphi(q)}\sum_{p}\log p\cdot e^{-xp}+O(x^{-(1/2)-\varepsilon})$$
$$=\frac{1}{x}\frac{\mu(q)}{\varphi(q)}+O(x^{-(1/2)-\varepsilon}).$$

Finally, since $\sum_{\gamma<0}\Gamma(K\rho)(x+iv)^{-K\rho}=O(1)$ and by Stirling's formula, we get an expression as is described in Theorem 2.

References

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