59. *Classification of Normal Congruence Subgroups of $G(\sqrt{q})$. II*

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This is continued from [0].

4. Here we treat the case of general level L. Any L can be written uniquely as $L=\prod L_p$ ($L_p \in \mathbf{L}$), where p runs over all primes and L_p is a power of p. Then we have the canonical isomorphism $H_q(L)\cong \prod_{p|L} H_q(L_p)$, where $p|L$ means $L_p\neq 1$. We regard $H_q(L_p)$ as a subgroup of $H_q(L)$ by this isomorphism. A set $\{N_1, \cdots, N_k\}$ of normal σ-subgroups of level L of $H_q(L)$ is called a *Z-complete set* of $H_q(L)$ if any normal σ-subgroup N of level L of $H_q(L)$ can be expressed as $N=N_iZ$ ($1\leqq i\leqq k$) by a σ-subgroup Z of $Z_q(L)$, where $Z_q(L)$ denotes the center of $H_q(L)$. Let $\mathfrak{S}=\{N_1, \cdots, N_k\}$ be a set of normal subgroups of $H_q(L)$ and K be a normal subgroup of $H_q(L)$. Then $\mathfrak{S}K$ denotes the set $\{N_1K, \cdots, N_kK\}$.

In order to define some normal σ-subgroups of level L of $H_q(L)$, we shall use the notation $[F, n; z]$ defined as follows. Let G_1 and G_2 be any two groups, and set $G=G_1\times G_2$. Let (F, n) be a pair of a normal subgroup F of G_1 and an element n of G_1. Let z be an element of the center of G_2. Then we set $[F, n; z]=F\times\langle z^2\rangle \cup nF\times z\langle z^2\rangle$.

For any integer $k\in \mathbf{N}$, put $L_k^*=\prod_{p\nmid k} L_p$. Suppose now that $q\neq 2$. When $L_2^*\neq 1$, the subset Z_q^* of $Z_q(L_2^*)$ is defined by $Z_q^*=\{z\in Z_q(q^{1/2})\prod_{p|L_{2q}^*}\{\pm I_p\} \mid \mathrm{ord}(z)=\text{even}\}$ (if $L_q=q^{1/2}$) or $\{z\in \prod_{p|L_2^*}\{\pm I_p\} \mid z\neq I\}$ (if $L_q\neq q^{1/2}$). Let us define the set $\mathfrak{S}_q(L)$ of subgroups of $H_q(L)$ by $\mathfrak{S}_q(L)=\{1\}$ (if $L_2=1$), $\{1, Q_1, [Q_1, B; z]\ (z\in Z_q^*)\}$ (if $L_2=2$), $\{1, E_2, Q_2\}$ (if $L_2=2^2$), $\{1, E_3\}$ (if $L_2=2^3$), $\{1, E_m, G_m^+, G_m^-, [F_m^+, X; z]\ (z\in Z_q^*), [F_m^+, -X; z]\ (z\in Z_q^*)\}$ ($X=\phi^{-1}(B_{m-1}C_{m-1}\cdot D_{m-4})$) (if $L_2=2^m$, $m\geqq 4$), where the groups of type $[F, n; z]$ are defined with respect to the decomposition $H_q(L)=H_q(2^m)\times H_q(L_2^*)$.

Theorem 4. *Assume that $q\neq 2$. Let L be any element of $\mathbf{L}$. Let $\mathfrak{S}_q(L)$ be the set defined above. Then a Z-complete set of $H_q(L)$ is given by the union of $\mathfrak{S}_q(L)$, $\mathfrak{S}_q(L)M$, $\mathfrak{S}_5(L)R_k^{(5)}$, $\mathfrak{S}_5(L)R_k^{(5)}M$, $\mathfrak{S}_5(L)S_k^{(5)}$, $\mathfrak{S}_5(L)S_k^{(5)}M$, where the sets multiplied by $R_k^{(5)}$ or $S_k^{(5)}$ appear only when $q=5$ and $L_5=5^k$ ($k\in \mathbf{N}$), and the sets multiplied by M appear only when $q\neq 3$ and $L_3=3$.*

Suppose now that $q=2$. When $L_2^*\neq 1$, set $Z_2^*=\{z\in \prod_{p|L_2^*}\{\pm I_p\} \mid z\neq I\}$. Let us define the set $\mathfrak{S}_2(L)$ of subgroups of $H_2(L)$ by $\mathfrak{S}_2(L)=\{1\}$ (if $L_2=2^{m-1/2}$ ($m\geqq 2$), 1, 2), $\{1, R_2, S_2, [\pm E_2^+, BC; z], [\pm E_2^+, BC^{-1}; z]\}$ (if $L_2=2^2$), $\{1, L_3^+, L_3^-, M_3^+, M_3^-, P_3, Q_3, S_3^+, S_3^-, [H_3^+, B_1C_1; z], [H_3^+, -B_1C_1; z], [H_3^+, B_1C_1^{-1}; z], [H_3^+, -B_1C_1^{-1}; z], [\pm \mathrm{L}_3^+, BC^{-1}; z], [\pm L_3^+, BC^{-1}D; z], [E_2^{3+}, BC; z], [E_2^{3+}, -BC; z]\}$ (if $L_2=2^3$), $\{1, L_m^+, L_m^-, M_m^+, M_m^-, N_m^+, N_m^-, O_m^+, O_m^-, [H_m^+, L; z], [H_m^+,$

$-L; z]$, $[H_m^+, M; z]$, $[H_m^+, -M; z]$, $[J_m^{++}, N; z]$, $[J_m^{++}, -N; z]$, $[J_m^{++}, O; z]$, $[J_m^{++}, -O; z]\}$ ($L=B_{m-2}C_{m-2}^{-1}$, $M=B_{m-2}C_{m-2}$, $N=B_{m-2}C_{m-2}D_{m-4}$, $O=B_{m-2}C_{m-2}^{-1}D_{m-4}$) (if $L_2=2^m$ ($m\geqq 4$)), where z runs over all elements of Z_2^*. The groups of type $[F, n; z]$ are defined with respect to the decomposition $H_2(L)=H_2(L_2)\times H_2(L_2^*)$.

Theorem 5. *Assume that $q=2$. Let L be any element of $\mathbf{L}$. When $L_2=2^{1/2}$, there are no subgroups of level L of $H_2(L)$. When $L_2\neq 2^{1/2}$, let $\mathfrak{S}_2(L)$ be the set defined above. Then a Z-complete set of $H_2(L)$ is $\mathfrak{S}_2(L)$ or $\mathfrak{S}_2(L)\cup\mathfrak{S}_2(L)M$ according as $L_3\neq 3$ or $L_3=3$ respectively.*

5. Now we consider odd groups. Let G be an odd normal congruence subgroup of level L of Γ. Then $G=N\cup(SX)N$, where $N=G\cap\Gamma^e$, $S=(0, -1; 1, 0)$ and $X\in\Gamma^e$.

Proposition. *Let N and X be an even normal subgroup of Γ and an even element respectively. Then the set $G=N\cup(SX)N$ is an odd normal subgroup of Γ if and only if the following two conditions are satisfied:*

(5.1) $X^{-1}PX\equiv P^{\sigma} \pmod N$ *for all* $P\in\Gamma^e$,

(5.2) $X^2\equiv -I \pmod N$.

By this proposition, the classification of all G reduces to the classification of all pairs (N, X) satisfying(5.1) and (5.2) with N of level L. Further, by the homomorphism $\Gamma^e\to H_q(L)$, the problem reduces to the classification of all pairs (N, X), where N is a normal σ-subgroup of level L of $H_q(L)$ and X is an element of $H_q(L)$, satisfying the following (5.3) and (5.4):

(5.3) $X^{-1}PX\equiv P^{\sigma} \pmod N$ for all $P\in H_q(L)$,

(5.4) $X^2\equiv -I \pmod N$.

We call such a pair (N, X) an *odd pair* of level L. Two odd pairs (N_1, X_1) and (N_2, X_2) are called *equivalent* if and only if $N_1=N_2$ and $X_1\equiv X_2$ (mod N_1). Then all G of level L corresponds one to one to all equivalence classes of odd pairs of level L of $H_q(L)$. First we treat the case that L is a power of a prime.

Theorem 6. *When $L=q^s$ ($s=m$ or $m-1/2$ ($m\in\mathbf{N}$)), all equivalence classes $(N, X \pmod N)$ of odd pairs of level L of $H_q(L)$ are the following:*

(1) $L=q^{1/2}$, $q\equiv 1 \pmod 4$: $(T_{(2)}^{(q)}, I)$, $(T_{(2)}^{(q)}, A)$.

(2) $L=3$: $(\pm S_1^{(3)}, I)$.

(3) $L=5$: $(\pm R_1^{(5)}, A)$, $(\pm S_1^{(5)}, I)$.

(4) $L=2$: (E_1, I), (E_1, B).

(5) $L=2^2$: (S_2, I), (S_2, B_1).

(6) $L=2^3$: $(S_3^+, \pm B_1)$.

When $L=p^m$ with p a prime $\neq q$, in particular in the case of $p=2$, there exist many equivalence classes of odd pairs. So we introduce a terminology "primitive". Let $N_1\supsetneqq N_2$ be two normal σ-subgroups of level L and let (N_2, X) be an odd pair. Then (N_1, X) is also an odd pair and we call it an extension of (N_2, X). An equivalence class of odd pairs is called *primitive* if it does not contain any odd pair which is an extension of other odd pairs. Now we define some elements of $H_q(p^m)$:

(1) If $p\neq 2$ and $(q/p)=1$, where (q/p) denotes the Legendre symbol, we set $X^{(q)}_{(p)m}=(0, b_m\sqrt{q}\,;\, -b_m\sqrt{q}, 0)$ where b_m is an integer such that $b_m^2 q\equiv 1 \pmod{p^m}$ and $b_m r\equiv 1 \pmod{p}$ with $1\leqq r<p/2$ $(r\in \boldsymbol{Z})$.

(2) If $p=2$ and $q\equiv 1 \pmod 8$, we set $X^{(q)}_{(2)m}=(0, b_m\sqrt{q}\,;\, -b_m\sqrt{q}, 0)$ where b_m is an integer such that $b_m^2 q\equiv 1 \pmod{2^{m+1}}$ and $b_m\equiv 1 \pmod 4$.

(3) If $p=2$, we set $Y^{(q)}_m=(0, \sqrt{q}\,;\, c_m\sqrt{q}, 0)$ where c_m is an integer such that $c_m q\equiv -1 \pmod{2^m}$.

Theorem 7. *When $L=p^m$ $(m\in N)$ with p a prime $\neq q$, all primitive equivalence classes $(N, X \pmod N)$ of odd pairs of level L of $H_q(L)$ are the following*:

(1) $L=p^m$ $(p\neq 2)$, $(q/p)=1$: $(1, \pm X^{(q)}_{(p)m})$.

(2) $L=2$: $(1, Y^{(q)}_1)$, $(Q^{(q)}_1, I)$.

(3-1) $L=2^2$, $q\equiv 1 \pmod 4$: $(1, \pm Y^{(q)}_2)$.

(3-2) $L=2^2$, $q\equiv 3 \pmod 4$: $(\pm I, Y^{(q)}_2 B_1 C_1)$.

(4-1) $L=2^3$, $q\equiv 1 \pmod 8$: $(1, \pm X^{(q)}_{(2)3})$, $(1, \pm X^{(q)}_{(2)3} D)$.

(4-2) $L=2^3$, $q\equiv 3 \pmod 8$: $(\pm E^{(q)}_3, Y^{(q)}_3 B_1 C_1)$, $(\pm E^{(q)}_3, Y^{(q)}_3 B_1 C_1 D)$.

(4-3) $L=2^3$, $q\equiv 5 \pmod 8$: $(F^{(q)}_3, \pm Y^{(q)}_3 B_2 C_2)$.

(4-4) $L=2^3$, $q\equiv 7 \pmod 8$: $(K^{(q)}_3, \pm Y^{(q)}_3 B_1 C_1)$.

(5) $L=2^m$ $(m\geqq 4)$, $q\equiv 1 \pmod 8$: $(1, \pm X^{(q)}_{(2)m})$, $(1, \pm X^{(q)}_{(2)m} D_{m-3})$, $(F^{(q)+}_m, \pm X^{(q)}_{(2)m} B_{m-1} C_{m-1} D_{m-4})$.

Second we consider the case of general level L. Then as in section 4, we have $L=\prod L_p$ and $H_q(L)=\prod H_q(L_p)$. For a normal σ-subgroup N of $H_q(L)$, the p-foot F_p of N is defined by $F_p=N\cap H_q(L_p)$. For an element X of $H_q(L)$, the p-component of X is denoted by X_p.

Theorem 8. *Let $L(\neq 1)$ be any element of $\boldsymbol{L}$. Let N, X, F_p and X_p be as above. Then the pair (N, X) is an odd pair of $H_q(L)$ of level L if and only if for each prime factor $p|L$ the pair (F_p, X_p) is an odd pair of $H_q(L_p)$ of level L_p.*

Reference

[0] T. Takagi: Classification of normal congruence Subgroups of $G(\sqrt{q})$. I. Proc. Japan Acad., **64A**, 167–169 (1988).