58. A Mathematical Theory of Randomized Computation. III

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Based on the results of earlier notes [5], we shall show that the category of randomized domains forms a cartesian closed monoid, which yields c.c.m. reduction calculi equivalent to type-free λ-calculi [1]. Then we shall axiomatize randomized domains, and show that our randomized domain theory is a natural probabilistic extension of Scott's theory. We also construct the reflexive graph model $\mathcal{G}\omega$ similar to Scott's $\mathcal{P}\omega$ [3].

11. The universal randomized domain $\mathcal{R}_\infty$. A reflexive object $\mathcal{R}_\infty$ in the c.c.c. ***CBL*** is constructed in quite the same way with the construction of $\boldsymbol{D}_\infty$ in Scott's theory [2]: Let $\mathcal{R}_0:=\mathcal{R}$ be any nontrivial domain in ***CBL*** and $\mathcal{R}_{n+1}:=[\mathcal{R}_n\to\mathcal{R}_n]$ $(\forall n\geq 0)$. Define $\varphi_n:\mathcal{R}_n\to\mathcal{R}_{n+1}$ and $\psi_n:\mathcal{R}_{n+1}\to\mathcal{R}_n$ by $\varphi_0(x):=\lambda y\in\mathcal{R}_0\cdot x$ $(\forall x\in\mathcal{R}_0)$, $\psi_0(y):=y(0)$ $(\forall y\in\mathcal{R}_1)$, $\varphi_{n+1}(x):=\varphi_n\circ x\circ\psi_n$ $(\forall x\in\mathcal{R}_{n+1})$, and $\psi_{n+1}(y):=\psi_n\circ y\circ\varphi_n$ $(\forall y\in\mathcal{R}_{n+2})$, for $\forall n\geq 0$.

Then $\langle\mathcal{R}_n,\psi_n\rangle_{n\in N}$ is a projective system of the domains in ***CBL***.

Let x_n denote the n-th coordinate of $x=(x_n)_{n=0}^\infty$ of the product $\prod_{n=0}^\infty\mathcal{R}_n$. Define the projective limit $\mathcal{R}_\infty$ by $\mathcal{R}_\infty:=\varprojlim\langle\mathcal{R}_n,\psi_n\rangle=\{x\in\prod_{n=0}^\infty\mathcal{R}_n|\forall n\in\boldsymbol{N},\ \psi_{n+1}(x_{n+1})=x_n\}$. Then $\mathcal{R}_\infty\in\boldsymbol{CBL}$ by (15)–(16). Define evaluation $\cdot$ in $\mathcal{R}_\infty$ by $x\cdot y:=\sup_n x_{n+1}(y_n)$. Then the evaluation $\cdot$ on $\mathcal{R}_\infty$ is positive order continuous. And we have:

(25) (i) (Extensionality) (a) $x\leq y\to\forall z\in\mathcal{R}_\infty,\ x\cdot z\leq y\cdot z$. (b) $x=y\to\forall z\in\mathcal{R}_\infty,\ x\cdot z=y\cdot z$. (ii) (Comprehension) *Define for* $f\in[\mathcal{R}_\infty\to\mathcal{R}_\infty]$, $\square f:=\sup_n\{\lambda y\in\mathcal{R}_n\cdot(f(y))_n\}$. *Then for* $\forall y\in\mathcal{R}_\infty$, $f(y)=\square f\cdot y$. (iii) (Reflexivity) $\mathcal{R}_\infty=[\mathcal{R}_\infty\to\mathcal{R}_\infty]$ *up to order isomorphism* (*and homeomorphism in the norm topology*).

Now that we have constructed a reflexive domain $\mathcal{R}_\infty$, the constructions of universal domains are straightforward: In fact, let X be the two point space of Boolean values and $\mathcal{R}_0:=\mathcal{H}(\ell^1(X))$ and construct $\mathcal{R}_\infty$ with this $\mathcal{R}_0$. Then we can define positive order continuous pairing function and associated selector functions. So $\mathcal{R}_\infty\times\mathcal{R}_\infty$ is a retract of $\mathcal{R}_\infty$ with these functions. Hence $\mathcal{R}_\infty$ is a *universal* domain of ***CBL*** and ***CBL*** forms a *cartesian closed monoid.*

The notion of band in our theory exactly corresponds to the notion of retract in Scott's theory. We recall the definitions:

(25) Let V be a BL. (i) A set $A\subset V$ is *solid* if $x\in A$, $y\in V$ and $|y|\leq|x|\Rightarrow y\in A$. (ii) An *ideal* of V is a solid vector subspace of V. (iii) An ideal B of V is a *band* of V if for $\forall$ non-empty $S\subset B$ possessing a supre-

mum sup S, $\sup S \in B$. (iv) A band B of V is a *projection band* if V is the direct sum of B and $B^{\perp}$, where $B^{\perp} := \{x \in V \mid |x| \wedge |y| = 0 \text{ for } \forall y \in B\}$. The projection $P: V \to B$ is called a *band projection*.

Then a band B of a KB-space V is a solid subspace which is a KB-space with the relative order and norm topologies and conversely. The band projection $P: V \to B$ is a retraction of V onto B in both order and norm topologies of V. Also $P: \mathcal{H}(V) \to \mathcal{H}(B)$. So every randomized domain can be obtained by some band projection from the *vniversal* domain $\mathcal{R}_\infty$.

12. The axiom systems for randomized domains. We present some axiom systems of randomized domains (cf. [4]):

Axiom system A. (Algebraic system)

Axiom A1 = Axiom 1 and Axiom A2 = Axiom 2 in § 2.

Axiom A3. *A randomized domain is the positive unit hemisphere* $\mathcal{H}(V)$ *of an algebraic KB-space* V.

Axiom A4. *operators between randomized domains are positive.*

Axiom system E. (Effective system)

Axiom E1 = Axiom 1 and Axiom E2 = Axiom 2.

Axiom E3. *A randomized domain is the positive unit hemisphere* $\mathcal{H}(V)$ *of a KB-space* V.

Axiom E4. *Operators between randomized domains are positive.*

Axiom E5. *A randomized domain has an effectively given countable basis for the order topology.*

Axiom system S. (Separable system)

Axiom S1 = Axiom 1 and Axiom S2 = Axiom 2.

Axiom S3. *A randomized domain is the positive unit hemisphere* $\mathcal{H}(V)$ *of a norm separable* σ*-order complete KB-space* V.

Axiom S4. *Operators between randomized domains are positive.*

Axiom S5. *A randomized domain has an effectively given countable basis for the order topology.*

13. Embedding of cpo's and randomized computability. Let Ω be a set, $\Omega_0 := \Omega \cup \{\perp\}$ where $\Omega \cap \{\perp\} = \phi$, and $(\Omega_0, \sqsubseteq)$ the complete poset (cpo) partially ordered by $\perp \sqsubseteq \perp \sqsubseteq x \sqsubseteq x$ for $\forall x \in \Omega$. Define the *strict* function $f_0: \Omega_0 \to \Omega_0$ for $\forall$ partial function $f: \Omega \to \Omega$ by $f_0(\perp) := \perp$ and $f_0(x) :=$ if $x \in$ domain (f) then $f(x)$ else $\perp$. Let $[\Omega_0 \to \Omega_0]$ be the cpo of $\forall$ Scott continuous functions $f_0: \Omega_0 \to \Omega_0$ partially ordered by $f_0 \sqsubseteq g_0$ iff $\forall x \in \Omega_0[f_0(x) \sqsubseteq g_0(x)]$.

Let V be the KB-space of all bounded measures on a measurable space $(\Omega, \mathcal{B}(\Omega))$. Now $\mu \in V$ can be viewed as a formal linear combination $\mu = \sum_{x \in \Omega} p_x 1_x$ of point masses 1_x, where $\forall x \in \Omega$, $\forall p_x \in \boldsymbol{R}$ and $\|\mu\| = \sum_{x \in \Omega} |p_x| < \infty$. Then Ω_0 and $[\Omega_0 \to \Omega_0]$ are naturally embedded into $\mathcal{H}(V)$ and $[\mathcal{H}(V) \to \mathcal{H}(V)]$ respectively by the *embedding* e defined by:

(26) $e(x) :=$ if $x \in \Omega$ then 1_x else 0 ($\forall x \in \Omega_0$), and

$e(f)e(x) :=$ if $fx \neq \perp$ then 1_{fx} else 0 ($\forall f \in [\Omega_0 \to \Omega_0]$).

$e(f)$ is uniquely extended to a positive operator $T_{e(f)}: V \to V$ by $T_{e(f)}(\mu) := \sum_{x \in \Omega} p_x \cdot 1_{f(x)}$, which are $\leq$-positive order continuous.

So if $f: \Omega \to \Omega$ is *partial computable*, then the operator T_f *defined* by f is $\leq$-*Scott continuous*. Thus we assert:

(27) (Church's thesis) *Let V and W be KB-spaces. An operator $T: \mathcal{H}(V) \to \mathcal{H}(W)$ is Scott continuous if there is a partial computable function f such that for $\forall \mu \in \mathcal{H}(V)$, $T(\mu) = \mu \circ f^{-1}$.*

14. The graph model $\mathcal{G}\omega$. We construct the reflexive graph model $\mathcal{G}\omega$ for randomized computation similar to Scott's $\mathcal{P}\omega$ [3]. Let ω be the set of natural numbers and $\mathcal{P}\omega := \{x | x \subset \omega\}$. First we fix the *coding* of binary rationals in the unit interval [0, 1] onto ω and the *coding* of an effectively given countable order dense set S in a given basic space X onto ω. We call an n-ary relation R in ω^n *single valued* if, for $\forall$ $(x_1, \cdots, x_{n-1})$, there is atmost one x_n s.t. $(x_1, \cdots, x_n) \in R$ and define the coding of the ordered pairs, finite sets and finite functions as follows:

(28) (i) $\langle x, y\rangle := (x+y)(x+y+1)/2+y$, $(\langle x, y\rangle)_1 := x$, and $(\langle x, y\rangle)_2 := y$ $(\forall x, y \in N)$. (ii) For $\forall n \in N$, define the coding of a finite set A_n by (a) $A_n = \phi \Leftrightarrow n=0$, and (b) a non-empty $A_n = \{x_1, \cdots, x_k\}$ with $x_1 < \cdots < x_k \Leftrightarrow n = \sum_{i \leq k} 2^{x} i$. (iii) A set A is said to be *single valued* if $\{(x, y) | \langle x, y\rangle \in A\}$ is a single valued relation. (iv) For $n \in N$, define the coding of the finite function ν_n by $\nu_n = \{(x_1, y_1), \cdots, (x_m, y_m)\}$ with $x_1 < \cdots < x_m \Leftrightarrow n = \{\langle x_1, y_1\rangle, \cdots, \langle x_m, y_m\rangle\}$.

Now we define our domain $\mathcal{G}\omega$ to be the set of single valued sets in $\mathcal{P}\omega$ partially ordered by the canonical ordering of functions $\leq$.

Then $(\mathcal{G}\omega, \leq)$ is a ccp with the Scott topology induced by $\leq$. Let $\mathcal{O}_n := \{\mu \in \mathcal{G}\omega | \nu_n \leq \mu\}$ for $\forall$ finite $\nu_n \in \mathcal{G}\omega$. Then $\{\mathcal{O}_n | n \in \omega\}$ forms a base for the Scott topology on $\mathcal{G}\omega$. Moreover we have:

(29) *Let $T: \mathcal{G}\omega \to \mathcal{G}\omega$.* (i) *If T is Scott continuous. Then T is monotone.* (ii) *T is Scott continuous iff $T(\mu) = \sup\{T(\nu) | \nu \leq \mu, \nu$ finite$\}$.*

Thus a Scott continuous $T: \mathcal{G}\omega \to \mathcal{G}\omega$ is determined by its value on the finite functions. So we can encode T as an element of $\mathcal{G}\omega$:

Define graph: $[\mathcal{G}\omega \to \mathcal{G}\omega] \to \mathcal{G}\omega$ and fun $\in [\mathcal{G}\omega \to \mathcal{G}\omega]$ by:

(30) (i) graph$(T) := \{\langle n, m\rangle | \nu_m = T(\nu_n)\}$ for $\forall T \in [\mathcal{G}\omega \to \mathcal{G}\omega]$.

(ii) fun$(u)(\mu) := \bigcup \{\nu_m | \exists \nu_n \leq \mu [\langle n, m\rangle \in u]\}$ $(\forall \mu \in \mathcal{G}\omega)$

for $\forall u \in \mathcal{G}\omega$. Then we have;

(31) (i) graph: $[\mathcal{G}\omega \to \mathcal{G}\omega] \to \mathcal{G}\omega$ *is Scott continuous.* (ii) *For* $\forall u \in \mathcal{G}\omega$, fun$(u) \in [\mathcal{G}\omega \to \mathcal{G}\omega]$. (iii) fun: $\mathcal{G}\omega \to [\mathcal{G}\omega \to \mathcal{G}\omega]$ *is Scott continuous.* (iv) *For* $\forall T \in [\mathcal{G}\omega \to \mathcal{G}\omega]$, fun (graph $(T)) = T$. (v) *For* $\forall u \in \mathcal{G}\omega$, graph (fun $(u)) \supset u$. (vi) (Reflexivity of $\mathcal{G}\omega$) $[\mathcal{G}\omega \to \mathcal{G}\omega] \simeq \mathcal{G}\omega$ (*order isomorphism*).

With Scott [3], the language LAMBDA has one primitive constant symbol 0, two unary function symbols $(x+1)$ and $(x-1)$, one binary function symbol $(u(x))$, and one ternary function symbol $(z \supset x, y)$ and one variable binding operator $(\lambda x \cdot \tau)$. The formation of the terms is defined in the obvious way. The semantics of LAMBDA in $\mathcal{G}\omega$ is defined as follows:

(32) $m[\![0]\!] := \{\langle 0, 1\rangle\}$, $m[\![\eta+1]\!] := \{\langle x+1, p\rangle | \langle x, p\rangle \in m[\![\eta]\!]\}$, $m[\![\eta-1]\!] := \{\langle x-1, p\rangle | \langle x, p\rangle \in m[\![\eta]\!]\}$, $m[\![\zeta \supset \eta, \theta]\!] := \lambda x \in \mathcal{G}\omega \cdot [\{\langle n, p\rangle | \langle \ell, p\rangle \in e_{\zeta}(x)$,

$\langle \ell, n\rangle \in m[\![\eta]\!]\} \cup \{\langle n, p\rangle \mid \langle \ell, p\rangle \in e_{\sim\zeta}(x), \langle \ell, n\rangle \in m[\![\theta]\!]\}]$, where $e_\zeta = \lambda x \in \mathcal{P}\omega \cdot \{\langle n, m\rangle \mid \langle \ell, m\rangle \in x, \langle \ell, 1\rangle \in m[\![\zeta]\!]\}$ and $e_{\sim\zeta} = \lambda x \in \mathcal{P}\omega \cdot \{\langle n, m\rangle \mid \langle \ell, m\rangle \in x, \langle \ell, 0\rangle \in m[\![\zeta]\!]\}$, $m[\![\eta(\mu)]\!] := \mathrm{fun}\, m[\![\eta]\!] m[\![\mu]\!]$, and $m[\![\lambda x \cdot \tau]\!] = \{\langle n, m\rangle \mid m[\![\tau]\!][\nu_n/\mathrm{x}] = \nu_m\}$.

Then the following definability theorem is obtained:

(33) (LAMBDA definability) *An operator* $T : \mathcal{P}\omega \to \mathcal{P}\omega$ *is computable iff* graph (T) *is* LAMBDA-*definable.*

References

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