57. An Elementary Proof of a Certain Transformation for an n-Balanced Hypergeometric $_3\Phi_2$ Series[†)]

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(Communicated by Kôsaku YOSIDA, M. J. A., June 14, 1988)

A rather elementary proof (based only upon the familiar Heine transformation for $_2\Phi_1$ is presented for an interesting generalization of a theorem asserting the symmetry in n and N of a function $f(n, N)$ which is defined in terms of an n-balanced basic (or q-) hypergeometric $_3\Phi_2$ series by Equation (5) below.

For real or complex q, $|q|<1$, let

$$(1) \qquad (\lambda ; q)_\mu=(\lambda ; q)_\infty/(\lambda q^\mu ; q)_\infty$$

for arbitrary λ and μ, where

$$(2) \qquad (\lambda ; q)_\infty=\prod_{j=0}^{\infty}(1-\lambda q^j).$$

The generalized basic (or q-) hypergeometric series defined by

$$(3) \qquad {}_{s+1}\Phi_s\left[\begin{matrix}\alpha_0, \cdots, \alpha_s ; \\ \\ \beta_1, \cdots, \beta_s ;\end{matrix} \; q, z\right]=\sum_{l=0}^{\infty}\frac{(\alpha_0 ; q)_l\cdots\alpha_s ; q)_l}{(\beta_1 ; q)_l\cdots(\beta_s ; q)_l}\frac{z^l}{(q ; q)_l} \qquad (|z|<1)$$

is said to be *n-balanced* if it terminates [that is, if at least one of the numerator parameters $\alpha_0, \cdots, \alpha_s$ is of the form q^{-N} $(N=0, 1, 2, \cdots)$], if $z=q$, and if (cf. Srivastava [2, p. 108])

$$(4) \qquad \beta_1\cdots\beta_s=q^{n+1}\alpha_0\cdots\alpha_s \qquad (n=0, 1, 2, \cdots),$$

it being understood, as usual, that no zeros appear in the denominator of (3). (Thus, for the sake of simplicity, a zero-balanced q-hypergeometric series is just called *balanced*; see also Askey and Wilson [1, p. 6].) We now recall a transformation formula for an n-balanced $_3\Phi_2$ series, which is contained in the following

Theorem (Srivastava [4, p. 109]). *Let n and N be arbitrary nonnegative integers. Then f(n, N) defined in terms of an n-balanced $_3\Phi_2$ series by*

$$(5) \qquad f(n, N)=\frac{(c ; q)_N(c/ab ; q)}{(c/a ; q)_N(c/b ; q)_N}\,{}_3\Phi_2\left[\begin{matrix}a, b, q^{-N} ; \\ \updownarrow \\ cq^n, abq^{1-N}/c ;\end{matrix} \; q, q\right]$$

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†) The work of the first author was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant A-7353.

Key words and phrases. Basic (or q-) hypergeometric series, n-balanced series, q-series identity, Jackson's sum, Pfaff-Saalschütz theorem, Sears' transformation, Heine's transformation, q-binomial theorem, analytic continuation.

is a symmetric function of n and N.

Two independent proofs of the theorem were given by Srivastava [4]. One of these proofs was based upon the following q-series identity due to Srivastava and Jain [5, p. 229, Equation (6.1)]:

$$(6)\qquad \sum_{l,m=0}^{\infty} \Omega_{l+m}(\lambda;q)_l(\mu;q)_m \frac{(\mu z)^l}{(q;q)_l}\frac{z^m}{(q;q)_m}=\sum_{n=0}^{\infty}\Omega_n(\lambda\mu;q)_n\frac{z^n}{(q;q)_n},$$

where $\{\Omega_n\}_{n=0}^{\infty}$ is a bounded sequence of complex numbers and the parameters λ and μ are essentially arbitrary, *and* upon Jackson's sum for a balanced ${}_3\Phi_2$ series:

$$(7)\qquad {}_3\Phi_2\left[\begin{matrix} a,\ b,\ q^{-N}; \\ \updownarrow\ q,\ q \\ c,\ abq^{1-N}/c; \end{matrix}\right]=\frac{(c/a;q)_N(c/b;q)_N}{(c;q)_N(c/ab;q)_N}\qquad (N=0,1,2,\cdots),$$

which provides a q-extension of the well-known Pfaff-Saalschütz theorem. The other proof of the theorem made use of Sears' transformation (cf. [3, p. 167, Equation (8.3)]; see also [1, p. 6, Equation (1.28)]):

$$(8)\qquad {}_4\Phi_3\left[\begin{matrix} \alpha,\ \beta,\ \gamma,\ q^{-N}; \\ \updownarrow\ q,\ q \\ \lambda,\ \mu,\ \nu; \end{matrix}\right]$$

$$=\frac{(\mu/\alpha;q)_N(\lambda\mu/\beta\gamma;q)_N}{(\mu;q)_N(\lambda\mu/\alpha\beta\gamma;q)_N}\,{}_4\Phi_3\left[\begin{matrix} \alpha,\ \lambda/\beta,\ \lambda/\gamma,\ q^{-N}; \\ \updownarrow\ q,\ q \\ \lambda,\ \alpha q^{1-N}/\mu,\ \alpha q^{1-N}/\nu; \end{matrix}\right],$$

which holds true when *each* ${}_4\Phi_3$ series is balanced, that is, when N is a nonnegative integer and [cf. Equation (4) with $n=0$]

$$\lambda\mu\nu=\alpha\beta\gamma q^{1-N}.$$

The object of this note is to present a rather elementary proof of the following slightly more general ${}_3\Phi_2$ transformation which, in fact, implies the assertion of the theorem fairly quickly:

$$(9)\qquad \frac{(cq^{\nu};q)_N(c/ab;q)_N}{(cq^{\nu}/a;q)_N(cq^{\nu}/b;q)_N}\,{}_3\Phi_2\left[\begin{matrix} a,\ b,\ q^{-N}; \\ \updownarrow q,\ q \\ cq^{\nu},\ abq^{1-N}/c; \end{matrix}\right]$$

$$=q^{-\nu N}\,{}_3\Phi_2\left[\begin{matrix} q^{1-\nu-N}/c,\ q^{-\nu},\ q^{-N}; \\ \updownarrow\ q,\ \dfrac{abq}{c} \\ aq^{1-\nu-N}/c,\ bq^{1-\nu-N}/c; \end{matrix}\right],$$

where N is a nonnegative integer, as before, but ν is unrestricted, in general.

Observe that the second member of (9) is symmetric in ν and N. Thus, in it special case when $\nu=n$ $(n=0,1,2,\cdots)$, the left-hand side of (9) leads immediately to the desired assertion that $f(n,N)$ defined by Equation (5) is a symmetric function of n and N.

Our proof of the transformation (9) is based upon such fundamental results as the familiar Heine transformation (cf. [2, p. 325, Theorem XVIII]; see also [6, p. 348, Equation (281)]):

$$(10) \qquad {}_2\Phi_1\left[\begin{matrix} a, b; \\ \updownarrow \\ c; \end{matrix}\; q, z\right] = \frac{(abz/c; q)_\infty}{(z; q)_\infty}\, {}_2\Phi_1\left[\begin{matrix} c/a, c/b; \\ \updownarrow \\ c; \end{matrix}\; q, \frac{abz}{c}\right]$$

and its obvious special case when $b=c$, *viz*

$$(11) \qquad \sum_{l=0}^{\infty} \frac{(a; q)_l}{(q; q)_l} z^l = \frac{(az; q)_\infty}{(z; q)_\infty} \qquad (|z|<1),$$

which is usually referred to as the q-binomial theorem. Indeed, if we replace c in (10) by cq^ν, we find for an arbitrary parameter ν that

$$(12) \qquad \frac{(z; q)_\infty}{(abz/c; q)_\infty}\, {}_2\Phi_1\left[\begin{matrix} a, b; \\ \updownarrow \\ cq^\nu; \end{matrix}\; q, z\right] = \frac{(abzq^{-\nu}/c; q)_\infty}{(abz/c; q)_\infty}\, {}_2\Phi_1\left[\begin{matrix} cq^\nu/a, cq^\nu/b; \\ \updownarrow \\ cq^\nu; \end{matrix}\; q, \frac{abz}{cq^\nu}\right].$$

For $|q|<1$ and $|z|<|cq^\nu/ab|$, each member of (12) can be expanded in (absolutely) convergent series of powers of z by means of (3) and (11). Equating the coefficients of z^N on the two sides of (12) thus expanded, and then appealing to the principle of analytic continuation, we are led easily to the general ${}_3\Phi_2$ transformation (9) (and hence also, as already pointed out, to the assertion of the theorem).

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