56. *Inclusion of Type III Factors Constructed from Ergodic Flows*

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1. Introduction. Given a common finite extension of two (conservative) ergodic flows we will construct a type III factor, its subfactor, and a normal conditional expectation (with finite index) such that the flows of weights of these factors are the given two ergodic flows.

In our previous announcement [3], we showed

Theorem 0. *Let M be a factor of type III with a subfactor N. Assume that $E: M\to N$ is a normal conditional expectation with* Index $E<\infty$ ([5]). *Let (T_t^M, X_M), (T_t^N, X_N) be the flows of weights of M and N respectively. Then there exists a (not necessarily ergodic) flow (T_t, X) satisfying:*
(i) *X is isomorphic to $X_M\times\{1,2,\cdots,m\}$ as a measure space for some positive integer m, $m\leqq$*Index E, *and simultaneously to $X_N\times\{1,2,\cdots,n\}$ for some positive integer n, $n\leqq$*Index E,
(ii) *the projection map π_M (resp. π_N) from X onto X_M (resp. X_N) intertwines T_t and T_t^M (resp. T_t and T_t^N).*

As mentioned at the beginning, we will obtain a converse of Theorem 0. Full details and further analysis will be published elsewhere.

2. Main theorem and remarks. Unless otherwise is stated, we will use the same notations as in [3]. All undefined terminologies can be found in [3] or references there.

At first we briefly recall main steps of Theorem 0. Let M_1 be the basic extension of $M\supseteqq N$ and $E_M: M_1\to M$ be the canonical conditional expectation constructed from E^{-1} in the usual way (see [4], [5]). Let ϕ be a faithful normal state on N. Setting $\psi=\phi\circ E\in M_*^+$ and $\chi=\psi\circ E_M\in(M_1)_*^+$, we looked at the inclusions

$$\tilde{M}_1=M_1\rtimes_{\sigma^\chi}\boldsymbol{R}\supseteqq\tilde{M}=M\rtimes_{\sigma^\psi}\boldsymbol{R}\supseteqq\tilde{N}=N\rtimes_{\sigma^\phi}\boldsymbol{R}$$

of continuous crossed products (all acting on $L^2(\tilde{M})$). Let $\{\theta_t\}_{t\in\boldsymbol{R}}$ be the dual action on these algebras. The measure space X_M is defined as the spectrum of the center $\tilde{M}\cap\tilde{M}'$, i.e., $L^\infty(X_M)=\tilde{M}\cap\tilde{M}'$. Notice that we are not interested in a measure itself on X_M but just a measure class. The space X_M and all other measure spaces in this paper are standard Borel. The space X_N is defined analogously. By the point-map realization theorem, θ_t induces an ergodic flow T_t^M (resp. T_t^N) on X_M (resp. X_N). The resulting flows (T_t^M, X_M), (T_t^N, X_N) are the flows of weights of M and N respectively. Set $Z=(\tilde{M}\cap\tilde{N}')\cap(\tilde{M}\cap\tilde{N}')'$, the center of the relative commutant. From (Z,θ_t)

we also get a (not necessarily ergodic) flow (X, T_t). The three flows are related to the each others as described in Theorem 0. A crucial observation here was that (T_t, X) can be identified with the flow arising from $\tilde{J}\{(\tilde{M}\cap\tilde{N}')\cap(\tilde{M}\cap\tilde{N}')'\}\tilde{J}=(\tilde{M}_1\cap\tilde{M}')\cap(\tilde{M}_1\cap\tilde{M}')'$ and θ_t, where $\tilde{J}$ is the modular conjugation on $L^2(\tilde{M})$.

Since the common extension is finite to one, we observe that there are only finitely many ergodic components. Each ergodic component itself is a common extension of the two flows of weights. Let $X_1, X_2, \cdots, X_k$ be the ergodic components in X, and assume that $\pi_M|_{X_i}$ and $\pi_N|_{X_i}$ are m_i to one and n_i to one respectively. (Hence, $m_1+m_2+\cdots+m_k=m$ and $n_1+n_2+\cdots+n_k=n$.)

Proposition 1. *If a ratio m_i/n_i is independent of i, then the product mn satisfies $mn\leqq$ Index E.*

Generally this ratio is not constant. However it is in the case that M is of type III_λ, $0<\lambda<1$. We thus obtain a slight strengthening of the result in [3]. An analogous result was independently obtained by Loi, [6].

As mentioned above X appeared as the spectrum of the abelian algebra $(\tilde{M}\cap\tilde{N}')\cap(\tilde{M}\cap\tilde{N}')'$.

Remark 2. The same construction as our proof of Theorem 0 works for the smaller abelian algebra $Z(\tilde{M})\vee Z(\tilde{N})$.

The reason why $(Z(\tilde{N})\subseteqq)Z(\tilde{M})\vee Z(\tilde{N})(\subseteqq Z(\tilde{M}\cap\tilde{N}'))$ works is that the modular conjugation $\tilde{J}$ satisfies

$$\tilde{J}(Z(\tilde{M})\vee Z(\tilde{N}))\tilde{J}=Z(\tilde{M}_1)\vee Z(\tilde{M}).$$

The center of $\tilde{M}\cap\tilde{N}'$ is useful for some purposes, but for our purpose in the present paper $Z(\tilde{M})\vee Z(\tilde{N})$ is more appropriate. For example, when (T_t, X) is constructed as in Remark 2, we get

$$L^\infty(X_M)\vee L^\infty(X_N)=L^\infty(X).$$

Note that via π_M and π_N we may regard $L^\infty(X_M)$ and $L^\infty(X_N)$ as subalgebras of $L^\infty(X)$.

Now we are ready to state our main result.

Theorem 3. *Let (F_t, Y), (S_t, Z) be conservative ergodic flows. Assume that a flow (T_t, X) is a common finite extension in the following sense: there exist finite to one maps π_Y and π_Z from X onto Y, Z respectively satisfying*

$$\pi_Y\circ T_t=F_t\circ\pi_Y \quad \text{and} \quad \pi_Z\circ T_t=S_t\circ\pi_Z$$

and $L^\infty(X)$ is generated by $L^\infty(Y)$ and $L^\infty(Z)$. Then we can construct factors M, N of type III with $M\supseteqq N$ and a normal conditional expectation from M onto N with finite index such that

(i) *(F_t, Y), (S_t, Z) are the flows of weights of M, N respectively,*

(ii) *the common extension constructed as in Theorem 0 and Remark 2 is exactly the given flow (T_t, X).*

3. Construction of type III factors. We will sketch a proof of Theorem 3. Let α be an ergodic transformation of type III_1 on a space (Ω_0, μ). We set

$$\Omega=\Omega_0\times X\times \boldsymbol{R}$$

equipped with a measure $m=\mu\otimes\nu\otimes e^u du$, where ν is a measure on X in the given measure class. Define (commuting) transformations $\tilde{\alpha}$ and $\tilde{T}_t$ ($t\in\boldsymbol{R}$) by

$$\begin{cases}\tilde{\alpha}(\omega, x, u)=\left(\alpha(\omega), x, u-\log\dfrac{d\mu\circ\alpha}{d\mu}(\omega)\right),\\ \tilde{T}_t(\omega, x, u)=\left(\omega, T_t x, u+t-\log\dfrac{d\nu\circ T_t}{d\nu}(x)\right).\end{cases}$$

Let G be the countable abelian group generated by $\tilde{\alpha}$ and $\tilde{T}_t$ ($t\in\Gamma$), where Γ is a countable dense subgroup in $\boldsymbol{R}$ (see [2]). By

$$R=L^\infty(\Omega)\rtimes G$$

we denote the von Neumann algebra (acting on its own standard Hilbert space H) obtained via the Krieger construction. Let $\mathcal{F}_Y$ (resp. $\mathcal{F}_Z$) be the smallest sub σ-algebra which makes the map $\tilde{\pi}_Y:(\omega, x, u)\in\Omega=\Omega_0\times X\times\boldsymbol{R}\to(\omega, \pi_Y(x), u)\in\Omega_0\times Y\times\boldsymbol{R}$ (resp. $\tilde{\pi}_Z$ defined analogously) measurable. We set

$$M_0=L^\infty(\Omega, \mathcal{F}_Y)\rtimes G\ (\subseteqq R),\qquad N=L^\infty(\Omega, \mathcal{F}_Z)\rtimes G\ (\subseteqq R).$$

Using the modular conjugation J on H we set

$$M=JM_0'J$$

so that we get $M\supseteqq R\supseteqq N$. Then it can be proved that the flows of weights of R, M_0 (or equivalently M), and N are the given three flows. Notice that M, N are factors while R is not.

Recall that von Neumann algebras constructed so far depend only on the measure class of ν. We can choose equivalent measures whose conditional probabilities with respect to $\mathcal{F}_Y$ and $\mathcal{F}_Z$ are constant on each ergodic component. Then we can construct normal conditional expectations

$$E_0: R\to M_0 \quad\text{and}\quad E_1: R\to N.$$

We can do this in such a way that $E_0^{-1}(1)$ is a scalar. Then

$$E_2=(E_0^{-1}(1))^{-1}JE_0^{-1}(J\cdot J)J$$

is a normal conditional expectation from M onto R, and the composition $E=E_1\circ E_2$ is a normal conditional expectation from M onto N.

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