54. On Bounded Positive Solutions of Quasilinear Elliptic Equations in R^n

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(Communicated by Kôsaku YOSIDA, M. J. A., June 14, 1988)

§1. Problem and result. Recently, Ni and Serrin [3]–[5] considered the existence and nonexistence problems of the ground state of the quasilinear elliptic equations and obtained several interesting results. In this paper, we consider the existence of the bounded positive entire solutions of the following two quasilinear elliptic equations

$$(\mathrm{P}^{\pm}) \qquad \operatorname{div}\left(Du\sqrt{1+|Du|^2}^{-1}\right)=\pm\phi(|x|)u^q$$

in Euclidean n-space R^n ($n\geqq 3$), where $q>1$ and Du denotes the gradient of u. By an entire solution of (P^+) [resp. (P^-)], we mean a function $u\in C^2(R^n)$ which satisfies (P^+) [resp. (P^-)] at every point of R^n. We assume that $\phi(r)$ satisfies

$$(\mathrm{A}) \qquad \phi\in C([0,\infty)),\ \phi\geqq 0 \text{ on } [0,\infty),\ \phi\not\equiv 0, \quad \text{and} \quad \int_0^\infty r\phi(r)dr<\infty.$$

In the previous paper [1, 2], we established the existence theorems of the bounded positive entire solutions of the semilinear elliptic equations, under the analogous assumption to (A). We emphasize that the methods used in [1] are also applicable for ($\mathrm{P}^{\pm}$) by using some estimates for the derivative.

Our theorem is stated as follows:

Theorem. *Suppose that the condition* (A) *holds. Then* (P^+) *and* (P^-) *have infinitely many bounded positive radial entire solutions, each of which tends to a positive constant as* $|x|\to\infty$.

The assertion of the theorem is easily followed by combining Lemmas 2 and 3 in the subsequent sections.

§2. Preliminaries. We seek positive radial solutions $u=u(|x|)$ of ($\mathrm{P}^{\pm}$). Put $r=|x|$, $p=|Du|$ and $A(p)=(1+p^2)^{-1/2}$, then by direct calculation, we have $p=|u'|$ and

$$\operatorname{div}(A(p)Du)=(A(p)u')'+(n-1)r^{-1}A(p)u',$$

where $'=d/dr$. Therefore, the problem is reduced to how to choose the initial values $\alpha>0$ so that the following ODE problems:

$$(\mathrm{Q}^{\pm}) \qquad \begin{cases} (1) & (A(p)u')'+(n-1)r^{-1}A(p)u'=\pm\phi(r)u^q, \qquad r>0, \\ (2) & u(0)=\alpha, \qquad u'(0)=0, \end{cases}$$

have positive solutions.

Now, by the argument similar to that of [4, Proposition 1], we introduce the integral equations which are equivalent to ($\mathrm{Q}^{\pm}$). Multiplying (1) by r^{n-1} and integrating it over $[0,r]$, we have

(3) $$A(p)u'(r)=\pm\int_0^r (s/r)^{n-1}\phi(s)u^q(s)ds.$$

Since $A(p)>0$, if (Q^+) [resp. (Q^-)] has positive solution $u(r)$, then it is clear that $u'(r)\geqq 0$ [resp. $u'(r)\leqq 0$] for all $r\geqq 0$. Therefore, it holds that

$$A(p)p=\int_0^r (sr^{-1})^{n-1}\phi(s)u^q(s)ds.$$

We denote $\Phi(p)=pA(p)$ and define Θ by

(4) $$\Theta[u](r)=\int_0^r (sr^{-1})^{n-1}\phi(s)u^q(s)ds.$$

If Θ satisfies $0\leqq\Theta<1$ for all $r\geqq 0$, then we have

(5) $$p=\Phi^{-1}(\Theta)=\Theta/\sqrt{1-\Theta^2},$$

where Φ^{-1} is the inverse of Φ. Noticing the sign of u', by an integration of (5), we obtain the following integral equations

(6) $$u(r)=\alpha\pm\int_0^r \Phi^{-1}(\Theta[u](t))dt.$$

Hence, we define the operators $\mathcal{F}^{\pm}: C([0,\infty))\to C([0,\infty))$ by

(7) $$\mathcal{F}^{\pm}[u](r)=\alpha\pm\int_0^r \Phi^{-1}(\Theta[u](t))dt.$$

We shall find the fixed point of $\mathcal{F}^{\pm}$ by applying Schauder-Tychonoff's fixed point theorem.

The following estimates for $\Theta[u](r)$ will be important in the subsequent sections.

Lemma 1. *Let the assumption* (A) *hold. If* $u\in C([0,\infty))$ *satisfies*

$$0\leqq u(r)\leqq C,\qquad r\geqq 0,$$

where C *is some positive constant, then the following two estimates hold*

(8) $$\Theta[u](r)\leqq mC^q,\qquad r\geqq 0,$$

(9) $$\int_0^r \Theta[u](t)dt\leqq (n-2)^{-1}\int_0^r s\phi(s)u^q(s)ds\leqq mC^q/(n-2),\qquad r\geqq 0,$$

where

(10) $$m=\max\left(\int_0^\infty \phi(r)dr,\int_0^\infty r\phi(r)dr\right).$$

For the sake of the convenience of the subsequent discussion, we introduce the following notation

(11) $$L(\xi)=\sup_{0<\theta<\xi}\Phi^{-1}(\theta)\theta^{-1}=1/\sqrt{1-\xi^2},\qquad 0<\xi<1.$$

§3. The equation (P^-). In this section, we consider the existence of the bounded positive entire solution of (P^-). The following lemma is closely related to Theorem 2.2 of [1].

Lemma 2. *If the initial value* α *is sufficiently small, then* (Q^-) *has a positive solution which is bounded and bounded away from zero.*

Proof. Choose a initial value $\alpha>0$ so small that

(12) $$0<m\alpha^q<1,$$

(13) $$1-L(m\alpha^q)(\alpha(n-2))^{-1}>0,$$

where m is the constant determined by (10).

Let $C([0,\infty))$ denote the locally convex space of continuous functions on $[0,\infty)$ with the topology of uniform convergence on every compact subinterval of $[0, \infty)$. Consider the closed convex subset

$$X=\{u \in C([0, \infty))\,;\, k(\alpha)\leqq u(r)\leqq \alpha,\ r\geqq 0\}$$

of $C([0, \infty))$, where,

$$k(\alpha)=\alpha-L(m\alpha^q)m\alpha^q/(n-2)>0. \tag{14}$$

Then by (11), (8) and (14), it is easily obtained that the operator $\mathcal{F}^-$ defined by (7) maps X into itself. From the Lipshitz continuity of $\Phi^{-1}(\Theta)=\Theta/(1-\Theta^2)^{1/2}$ on $0\leqq\Theta\leqq m\alpha^q$ (<1) and Ascoli-Arzelà's theorem, it is easily seen that $\mathcal{F}^-$ is a completely continuous operator. Therefore, $\mathcal{F}^-$ has a fixed point $u \in X$. Q.E.D.

§ 4. The equation (P^+). In this section, we deal with the equation (P^+). The following lemma corresponds to Theorem 2.5 of [1].

Lemma 3. *Let the assumption* (A) *hold. If the initial value α is sufficiently small, then* (Q^+) *has a positive solution which is bounded and bounded away from zero.*

Proof. Take a positive constant β so that

$$0<m\beta^q<1, \tag{15}$$

where m is the constant determined by (10). Next, choose an initial value $\alpha>0$ so small that

$$\alpha^{1-q}\geqq\beta^{1-q}+(q-1)mL(m\beta^q)(n-2)^{-1}. \tag{16}$$

We note that $\alpha\leqq\beta$. We define a function $B(r)$ by

$$B(r)=\left(\alpha^{1-q}-(q-1)L(m\beta^q)(n-2)^{-1}\int_0^r t\phi(t)\,dt\right)^{1/1-q}, \tag{17}$$

which is a solution of the following integral equation

$$B(r)=\alpha+L(m\beta^q)(n-2)^{-1}\int_0^r s\phi(s)B^q(s)ds. \tag{18}$$

It is readily seen that $\alpha\leqq B(r)\leqq\beta$ for all $r\geqq 0$. We now introduce

$$Y=\{u \in C([0, \infty))\,;\, \alpha\leqq u(r)\leqq B(r),\ r\geqq 0\},$$

which is a closed convex subset of locally convex space $C([0, \infty))$. By virtue of (15) and Lemma 1, $\mathcal{F}^+$ is well defined on Y and an inequality, $\alpha\leqq\mathcal{F}^+[u](r)$, is satisfied. Consequently, we obtain

$$\begin{aligned}\mathcal{F}^+[u](r)&\leqq\alpha+L(m\beta^q)\int_0^r \Theta[u](t)dt\\ &\leqq\alpha+\frac{L(m\beta^q)}{n-2}\int_0^r s\phi(s)u^q(s)ds\\ &\leqq\alpha+\frac{L(m\beta^q)}{n-2}\int_0^r s\phi(s)B^q(s)ds\\ &=B(r), \qquad r\geqq 0,\end{aligned}$$

by (11), (9) and (18). Hence, we have shown that $\mathcal{F}^+ : Y\to Y$. It is easily seen that $\mathcal{F}^+$ is a completely continuous operator. Therefore $\mathcal{F}^+$ has a fixed point $u \in Y$, which satisfies (Q^+) and $\alpha\leqq u(r)\leqq\beta$ for all $r\geqq 0$. Q.E.D.

Concluding remark. We have considered the concrete equation of the form: $A(p)=(1+p^2)^{-1/2}$, however this restriction can slightly be weakened. For instance, if $A(p)$ satisfies $A \in C^1([0, \infty))$ with $A(0)>0$, then it is easy to see that $\Phi(0)=0$ and $\Phi'(p)>0$ for sufficiently small $p>0$. Here $\Phi(p)=pA(p)$.

Hence $\Phi^{-1}(\cdot)$ is locally Lipshitz continuous on an interval $[0, \varepsilon]$ for some $\varepsilon > 0$. Therefore, if we choose an initial value sufficiently small, then we have

$$\int_0^r \Phi^{-1}(\Theta[u](t))dt \leqq L(\varepsilon) \int_0^r \Theta[u](t)dt,$$

where $\Theta[u](r)$ is defined as in section 2 and $L(\varepsilon) = \sup\{\Phi^{-1}(\theta)/\theta\,;\, 0 < \theta < \varepsilon\}$. This implies that the method used in the proof of lemmas can be applicable.

The author would like to express his sincere gratitude to Prof. Manabu Naito and Prof. Shoji Yotsutani for their helpful comments concerning this work.

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