53. *Kähler Diffusion Processes Associated with the Bergman Metric and Domains of Holomorphy*

By Setsuo TANIGUCHI
Department of Applied Science, Faculty of Engineering, Kyushu University

(Communicated by Kôsaku YOSIDA, M. J. A., June 14, 1988)

1. In this paper, we will see that the conservativeness of the minimal diffusion process associated with the Bergman metric of a bounded domain in C^n implies that the domain is a domain of holomorphy and will give a necessary and sufficient condition for the diffusion process to be conservative.

Since Oka established that every pseudoconvex domain in C^n is a domain of holomorphy, it has been one of central topics in complex analysis to characterize domains of holomorphy by the geometric properties of the domain. It was seen by Bremermann ([1]) that a bounded domain in C^n is a domain of holomorphy if the Bergman metric of the domain is complete. Diederich and Pflug ([2]) showed that a bounded domain D with a complete Kähler metric is a domain of holomorphy under the restriction that $\bar{D}^0=D$ (see also Grauert [3]). Since the geometric aspects of a Kähler manifold reflects on the behaviour of the minimal diffusion process corresponding to the Kähler metric, we can expect to characterize domains of holomorphy by suitable properties of the diffusion process. In fact, under the same restriction as the one made by Diederich-Pflug, we will establish that D is a domain of holomorphy whenever the Kähler diffusion process associated with the Bergman metric is conservative.

Let $D\subset C^n$ be a bounded domain and $H^2(D)$ be the set of square integrable holomorphic functions on D. The Bergman kernel function $K(z)$, $z\in D$, is defined by

$$K(z)=\sup\{|f(z)|^2/\|f\|^2:\ f\in H^2(D)\backslash\{0\}\},$$

where $\|f\|^2=\int_D |f(z)|^2 dm(z)$, dm being the Lebesgue measure on D. It is well known that $K(z)>0$ and $\log K(z)$ is strictly plurisubharmonic. The Bergman metric β of D is by definition given by

$$\beta=\textstyle\sum_{i,j=1}^n (\partial^2 \log K/\partial z^i \partial \bar{z}^j) dz^i d\bar{z}^j.$$

We denote by $\boldsymbol{B}(D)=(Z_t,\ \zeta,\ P_z)_{z\in D}$ the minimal diffusion process generated by $\Delta/2$, where ζ is the life time and Δ is the Laplace-Beltrami operator corresponding to the Bergman metric β (for details, see [4], [8]). We have obtained

Theorem 1. *Let a bounded domain $D\subset C^n$ satisfy that $\bar{D}^0=D$. Then D is a domain of holomorphy whenever $\boldsymbol{B}(D)$ is conservative, i.e., $P_z(\zeta=+\infty)=1$ for every $z\in D$.*

To apply our criterion, it is important to know when the diffusion process $\boldsymbol{B}(D)$ is conservative. Since the definition of the Bergman metric is so abstract, it is very difficult, in general, to get any information on curvatures although most stochastic geometrical criteria for conservativeness are given in terms of curvatures. However, noting that the Bergman metric is given as a complex Hessian matrix of $\log K(z)$, we have the following necessary and sufficient condition for $\boldsymbol{B}(D)$ to be conservative.

Theorem 2. *It holds that*

$$P_z[\{\zeta=+\infty\}\varDelta\{\limsup_{t\uparrow\zeta} K(Z_t)=+\infty\}]=0 \qquad \text{for every } z\in D,$$

where $A\varDelta B=(A\backslash B)\cup(B\backslash A)$. *In particular,* $\boldsymbol{B}(D)$ *is conservative if and only if it is satisfied that*

$$P_z[\{\zeta<+\infty\}\cap\{\limsup_{t\uparrow\zeta} K(Z_t)<+\infty\}]=0 \qquad \text{for every } z\in D.$$

The following is an immediate consequence of Theorem 2.

Corollary. $\boldsymbol{B}(D)$ *is conservative provided that* $K(z)$ *is exhaustive, i.e., for every sequence* $\{z_k\}\subset D$ *with* $z_k\to z^*\in\partial D$, $\limsup_{k\to\infty} K(z_k)=+\infty$.

In our knowledge, all criteria for completeness with respect to the Bergman metric require the assumption that the Bergman kernel function $K(z)$ is exhaustive (for example, see [5], [6], [7]). Moreover, Kobayashi ([5]) conjectured that the completeness implies the exhaustion property of $K(z)$.

2. Sketch of proofs. Assume that $\bar{D}^0=D$. Suppose that D is not a domain of holomorphy. Let Ω be an envelope of holomorphy of D. Since $\bar{D}^0=D$, for each $z^*\in\partial D\cap\Omega$, there are open sets U and V such that $z^*\in U\subset\subset\Omega$ and $V\subset\subset U\backslash\bar{D}$. Bremermann ([1]) has shown that the Bergman metric β can be extended to a Kähler metric g on Ω. Applying Stroock-Varadhan's support theorem ([9]) we can conclude that the Kähler diffusion process $(Y_t, \eta, Q_y)_{y\in\Omega}$ on Ω corresponding to the Kähler metric g enjoys that for each $z\in U\cap D$,

$$Q_z\{Y_t \text{ hits } V \text{ in finite time before leaving } U\}>0,$$

which means that $\boldsymbol{B}(D)$ is not conservative. Thus we have verified Theorem 1.

To see Theorem 2, note that $\varDelta\log K(z)=n$, $z\in D$. By Itô's formula and a standard time change argument, we have, under P_z,

$$\log K(Z_t)=\log K(z)+W(\Phi_t)+(nt/2), \tag{2.1}$$

where $W(t)$ is an $(\boldsymbol{R})^1$-valued Brownian motion with $W(0)=0$,

$$\Phi_t=\int_0^t \sum_{i,j=1}^n [\beta^{i\bar{j}}(\partial\log K/\partial z^i)(\partial\log K/\partial\bar{z}^j)](Z_s)ds$$

and $(\beta^{i\bar{j}})_{1\leq i,j\leq n}$ is the inverse matrix of $(\partial^2\log K/\partial z^i\partial\bar{z}^j)_{1\leq i,j\leq n}$. Then we conclude

$$\{\zeta=+\infty,\ \limsup_{t\uparrow\zeta}\log K(Z_t)<+\infty\}\subset\{\lim_{t\uparrow\zeta} W(\Phi_t)=-\infty\},$$

$$\{\zeta<+\infty,\ \limsup_{t\uparrow\zeta}\log K(Z_t)=+\infty\}\subset\{\liminf_{t\uparrow\zeta}\log K(Z_t)=-\infty\},$$

taking advantage of (2.1) and the fact that $\limsup_{t\uparrow\infty} W(t)=+\infty$ and $\liminf_{t\uparrow\infty} W(t)=-\infty$. Since $\inf\{K(z): z\in D\}>0$, these implies that the identity in Theorem 2 holds.

3. Remarks. 1) Without the assumption that $\bar{D}^0=D$, the assertion of Theorem 1 does not hold, in general. For example, let D^* be a bounded domain in C^n such that $\boldsymbol{B}(D^*)$ is conservative and A be an analytic set of D^* with $\dim A \leqq n-2$. Then, by Riemann's extension theorem, $D=D^*\backslash A$ is not a domain of holomorphy. However, $\boldsymbol{B}(D)$ is still conservative because $\boldsymbol{B}(D)$ is the restriction of $\boldsymbol{B}(D^*)$ to D and $\boldsymbol{B}(D^*)$ never hits analytic sets of dimension $\leqq n-1$.

2) The argument used in the proof of Theorem 2, if slightly modified, still works in the case that M is a complex manifold possessing a Bergman metric (for the definition, see [5]). Thus the assertion in Theorem 2 with $f\wedge\bar{f}/K'$ for K is true in such a case, where f is a non-zero square integrable holomorphic $(n, 0)$-form on M, K' is the kernel form of M (see [5]) and $f\wedge\bar{f}/K'$ is a unique $g \in C^\infty(M)$ such that $f\wedge\bar{f}=gK'$.

References

[1] H. Bremermann: Holomorphic continuation of the kernel function and the Bergman metric in several complex variables. Lectures on Functions of a Complex Variable. Univ. of Michigan Press, Michigan, pp. 349–383 (1955).

[2] K. Diederich and P. Pflug: Über Gebiete mit vollständiger Kähler Metrik. Math. Ann., **257**, 191–198 (1981).

[3] H. Grauert: Charakterisierung der Holomorphiegebiete durch die vollständige Kählerische Metrik. ibid., **131**, 38–75 (1956).

[4] N. Ikeda and S. Watanabe: Stochastic Differential Equations and Diffusion Processes. Kodansha, Tokyo; North-Holland, Amsterdam (1981).

[5] S. Kobayashi: Geometry of bounded domains. Trans. Amer. Math. Soc., **92**, 267–290 (1959).

[6] T. Ohsawa: A remark on the completeness of the Bergman metric. Proc. Japan Acad., **57A**, 238–240 (1981).

[7] P. Pflug: Various applications of existence of well growing holomorphic functions. Functional Analysis, Holomorphy and Approximation Theory (ed. J. A. Barroso). North-Holland, Amsterdam (1982).

[8] D. W. Stroock and S. R. S. Varadhan: Multidimensional Diffusion Processes. Springer-Verlag, Berlin (1979).

[9] ——: On the support of diffusion processes with applications to the strong maximum principle. Proc. Sixth Berkeley Symp. Math. Statist. Prob. III, Univ. California Press, Berkeley, pp. 333–359 (1972).