52. *Boundary Values of HBD-functions on Harmonic Boundaries of Riemann Surfaces*

By Hisashi ISHIDA
Kyoto Sangyo University

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Introduction. Every harmonic measure on a compact bordered Riemann surface takes a constant value on each component of the border. While for general open Riemann surfaces R, Kusunoki [6] and Kusunoki-Mori [7] showed that every harmonic measure takes a constant value quasi-everywhere (resp. almost everywhere) on each connected component of the Kuramochi (resp. Royden) boundary of R.

In this paper, we shall introduce some new classes of HBD-functions containing harmonic measures and describe the results on the boundary values of the functions in these classes at the connected components of (Royden) harmonic boundary of R. The details will be published in another paper [5] together with related topics.

1. In the following we assume that R is an open Riemann surface which admits the Green functions. We denote by $BD(R)$ the class of (real) bounded continuous Dirichlet functions on R and by $HBD(R)$ the class of harmonic functions in $BD(R)$. We consider a topology in $BD(R)$. Let $\{f_n\}$ be a sequence of functions on R and f a function on R. We say that $\{f_n\}$ converges to f on R in BD-topology and write $f=BD\text{-}\lim_{n\to\infty} f_n$ if $\{f_n\}$ is uniformly bounded on R, $\{f_n\}$ converges to f uniformly on every compact set of R and $\lim_{n\to\infty} D_R(f_n-f)=0$, where $D_R(h)$ is the Dirichlet integral of $h\in BD(R)$ on R. Then we know that both $BD(R)$ and $HBD(R)$ are BD-complete. Let R^* be the *Royden compactification* and $\varDelta$ the (Royden) *harmonic boundary* of R. Then, we know that every function in $BD(R)$ can be continuously extended to R^* (cf. [4], [8]).

Definition. We say that a function u on an open Riemann surface R is an HBD_{c0}-(resp. BD_{c0}-) function if $u\in HBD(R)$ (resp. $BD(R)$) and

$$(du, {}^*dh)_R=\iint_R du\wedge dh=0$$

for any BD-function h, where $^*dh$ is the conjugate differential of dh.

The orthogonal condition above implies that $du\in\varGamma_{c0}$ (cf. [3]). Then we see that both $BD_{c0}(R)$ and $HBD_{c0}(R)$ are also BD-complete. Moreover $BD_{c0}(R)$ enjoys a lattice property. That is, for every f and g in $BD_{c0}(R)$, $\max(f, g)$ and $\min(f, g)$ belong to $BD_{c0}(R)$. Using the harmonic boundary, we have the following

Theorem 1. *Suppose that u is an HBD-function on R such that the range of values $\{u(p)\,;\,p\in\varDelta\}$ is a finite set. Then u is an HBD_{c0}-function on R.*

2. A bounded harmonic function u on R is called a *generalized harmonic measure* on R if the greatest harmonic minorant of u and $1-u$ vanishes identically on R. Then we have the following characterization of the generalized harmonic measures with finite Dirichlet integral.

Theorem 2. *For a non-constant HBD-function u on R, u is a generalized harmonic measure on R if and only if* $\{u(p); p \in \Delta\}=\{0, 1\}$.

Let $HM(R)$ be the totality of HBD-functions u such that $du \in \Gamma_{hm}$ (cf. [3]), that is, the class of HBD-functions generated by harmonic measures.

Definition. An HBD-function u on R is called an $\widehat{HM}$-function if $u= BD\text{-}\lim_{n\to\infty} u_n$, where each u_n is a (real) linear combination of generalized harmonic measures with finite Dirichlet integral on R.

By Theorems 1 and 2 we can prove the following

Theorem 3. $HM(R)\subset\widehat{HM}(R)\subset HBD_{c0}(R)$.

If R is a Riemann surface of finite genus, then $HM(R)=\widehat{HM}(R) =HBD_{c0}(R)$. However we have an example of a Riemann surface R of infinite genus such that $HM(R)\subsetneqq\widehat{HM}(R)$ (cf. [1]).

Moreover we can show the following geometrical condition in order that the class $\widehat{HM}(R)$ degenerates to the class of constant functions.

Theorem 4. *There are no non-constant $\widehat{HM}$-functions on R if and only if Δ is connected.*

Let p_0 be a point of R. We know that every HBD-function has a radial limit along almost every Green's line (that is, except for a set of Green's lines of Green measure zero) issuing from p_0 (cf. [8, p. 203]). For each ideal boundary component e of Kerékjártó-Stoïlow compactification of R, we set $\Delta_e=(\bigcap_U \overline{(U\cap R)})\cap\Delta$, where U represents the neighborhood of e and the closure is taken in R^*. Using the method of Green's lines, we can prove that every HM-function on R takes a constant value μ-almost everywhere on each Kerékjártó-Stoïlow component Δ_e of Δ, where μ is the harmonic measure of R^*-R with respect to p_0 (cf. [7]). For $\widehat{HM}$-functions we get the following

Theorem 5. *Every $\widehat{HM}$-function u on R has a constant value on each connected component of Δ except for a set of μ-measure zero.*

We know that for every HM-function u there exist a canonical exhaustion $\{R_n\}$ and $u_n \in HM(R_n)$ such that $\lim_{n\to\infty} D_{R_n}(u_n-u)=0$ (cf. [3]). In contrast to the fact above, we have the following

Theorem 6. *Suppose that $u\in\widehat{HM}(R)$, then there exist a (not necessarily canonical) exhaustion $\{R_n\}$ and $u_n \in HM(R_n)$ such that* $\lim_{n\to\infty} D_{R_n}(u_n-u)=0$.

3. We consider Riemann surfaces for which the dimension of the space of HBD_{c0}-functions is finite. For the HBD_{c0}-functions and $\widehat{HM}$-

functions on such a Riemann surface R, we have the following

Theorem 7. *If R is a Riemann surface for which* $\dim HBD_{c0}(R) = n\ (<\infty)$, *then*

(1) *Δ consists of n components*

and

(2) $\widehat{HM}(R) = HBD_{c0}(R) = \{u \in HBD(R)$; *$u$ takes a constant value on each connected component of Δ*$\}$.

We note that there exists a Riemann surface for which $\dim HBD_{c0}$ is finite. In fact, we can prove the following

Theorem 8. *If R is a complete n-sheeted branched covering surface of the unit disk, then* $\dim HBD_{c0}(R)$ *is at most n. Moreover for any integers $1 \leqq m \leqq n$, there exists a covering Riemann surface R on which* $\dim HM(R) = m$ *and* $\dim \widehat{HM}(R) = \dim HBD_{c0}(R) = n$.

For the proof of the theorem, we use the fact the *class of BD_{c0}-functions forms an algebra.*

References

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