44. On a Weak Generalization of the Fundamental Theorem of the Theory of Curves or Hypersurfaces[†]

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0. Introduction. Let us consider the Euclidean space $\boldsymbol{R}^3$ and a surface with an analytic representation ${}^t(f_1, f_2, f_3)=f(x_1, x_2)$. Then, for this, we have first fundamental quantities $K_{ij}(j^1_x(f))=p_i\cdot p_j$ $(1\leq i, j\leq 2)$ and second fundamental quantities $L_{ij}(j^2_x(f))=|p_{ij}, p_1, p_2|/\sqrt{K_{11}K_{22}-(K_{12})^2}$ $(1\leq i, j\leq 2)$ where the dot means the canonical inner product in $\boldsymbol{R}^3$ and $p_i={}^t(\partial f_1/\partial x_i, \partial f_2/\partial x_i, \partial f_3/\partial x_i)$, $p_{ij}={}^t(\partial^2 f_1/\partial x_i\partial x_j, \partial^2 f_2/\partial x_i\partial x_j, \partial^2 f_3/\partial x_i\partial x_j)$. K_{ij} (resp. L_{ij}) is considered as a function on the 1-jet space $J^1(\boldsymbol{R}^2, \boldsymbol{R}^3)$ (resp. the 2-jet space $J^2(\boldsymbol{R}^2, \boldsymbol{R}^3)$). For the above particular f, if we set $\lambda_{ij}(x)=K_{ij}(j^1_x(f))$ and $\eta_{ij}(x)=L_{ij}(j^2_x(f))$, then we get a system of differential equations $P: K_{ij}-\lambda_{ij}=0$ $(1\leq i, j\leq 2)$, $L_{ij}-\eta_{ij}=0$ $(1\leq i, j\leq 2)$.

Let Γ be the pseudogroup generated by local isometries on the Euclidean space $\boldsymbol{R}^3$. Then the fundamental theorem of the theory of surfaces means that any solution s of P is written by $s=\sigma\circ f$ for some $\sigma\in\Gamma$.

A similar fact holds for curves in $\boldsymbol{R}^3$ with an analytic representation ${}^t(f_1, f_2, f_3)=f(t)$ using the torsion and the curvature of f.

The purpose of this note is to generalize the above stated facts to any local immersion $f: \boldsymbol{R}^n\to\boldsymbol{R}^m$ $(n<m)$ and any pseudogroup Γ of finite type on $\boldsymbol{R}^m$ in a generic situation for f and Γ. The smoothness is always assumed to be of class C^ω.

1. Statement of the results. Let $J^k(n, m)$ be the space of k-jets of local maps of $\boldsymbol{R}^n$ to $\boldsymbol{R}^m$. If $n<m$, then $\tilde{J}^k(n, m)$ means the space of k-jets of local immersions and if $n\geq m$, $\tilde{J}^k(n, m)$ means the space of k-jets of local submersions. In both cases, $\tilde{J}^k(n, m)$ is open and dense in $J^k(n, m)$.

Let Γ be a pseudogroup on $\boldsymbol{R}^m$. Then Γ is lifted to a pseudogroup $\Gamma^{(k)}_n$ on $\tilde{J}^k(n, m)$ by $\phi^{(k)}(j^k_x(f))=j^k_x(\phi\circ f)$.

A vector field X on $\boldsymbol{R}^m$ is called a Γ-vector field if the local 1-parameter group of local transformations on $\boldsymbol{R}^m$ generated by X is contained in Γ. Let $\mathcal{L}$ denote the sheaf on $\boldsymbol{R}^m$ of germs of Γ-vector fields. Then $\mathcal{L}$ is also lifted to a sheaf $\mathcal{L}^{(k)}_n$ on $\tilde{J}^k(n, m)$. $(\mathcal{L}^{(k)}_n)_p$ (resp. $\mathcal{L}_z$) means the stalk of $\mathcal{L}^{(k)}_n$ (resp. $\mathcal{L}$) on $p\in\tilde{J}^k(n, m)$ (resp. $z\in\boldsymbol{R}^m$).

Definition 1.1. A function ϕ on a neighbourhood of a point $p\in\tilde{J}^k(n, m)$ is called a differential invariant of Γ at p if $X\phi=0$ for any $X\in(\mathcal{L}^{(k)}_n)_p$.

Let $\{\phi_1, \cdots, \phi_r\}$ be a maximal family of differential invariants of Γ at $j^k_x(f)$ such that the differentials $d\phi_1, \cdots, d\phi_r$ are linearly independent at $j^k_x(f)$

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where $j_x^k(f) \in \tilde{J}^k(n, m)$.

Definition 1.2. Γ is said to be k-regular at (x, f) if the family $\{\phi_1, \cdots, \phi_r\}$ is also a maximal family of differential invariants of Γ at any point $p \in U$ for some neighbourhood U of $j_x^k(f)$. Then the family is called a fundamental system of differential invariants of Γ at $j_x^k(f)$. Γ is said to be regular at (x, f) if it is k-regular at (x, f) for any integer $k \geq 0$.

Assume that Γ is k-regular at (x_0, f) and let $\{Y_1^k, \cdots, Y_{m_k}^k\}$ be a fundamental system of differential invariants of Γ at $j_{x_0}^k(f)$. We set $\lambda_i^k(x) = Y_i^k(j_x^k(f))$. Then we have a system of differential equations $P^k : Y_i^k = \lambda_i^k$ $(i=1, \cdots, m_k)$ around $j_{x_0}^k(f)$. P^k is called the Γ-orbit system at (x_0, f).

Definition 1.3. A system of differential equations Φ^l given around $j_{x_0}^l(f)$ is said to be Γ-automorphic if (1) f belongs to the solution space $\mathcal{S}(\Phi^l)$ of Φ^l, (2) for any $\sigma \in \Gamma$ which is near to the identity and for any $s \in \mathcal{S}(\Phi^l)$, we have $\sigma \circ s \in \mathcal{S}(\Phi^l)$ and (3) for any $s \in \mathcal{S}(\Phi^l)$ near to f, there exists $\sigma \in \Gamma$ such that $s = \sigma \circ f$.

Then our main theorem is

Theorem 1.1. *Let Γ be a pseudogroup on $\boldsymbol{R}^m$ and assume that* $\dim \mathcal{L}_z < \infty$ *at $z \in \boldsymbol{R}^m$ and that Γ is regular at (x, f) where $f(x) = z$. Then for a sufficiently large integer k, P^k is Γ-automorphic.*

2. Proof. Let π_k^{k+h} denote the natural projection of $\tilde{J}^{k+h}(n, m)$ onto $\tilde{J}^k(n, m)$ and let α^k (resp. β^k) denote the natural projection of $\tilde{J}^k(n, m)$ onto $\boldsymbol{R}^n$ (resp. $\boldsymbol{R}^m$). Let $\mathcal{U}^k$ be a neighbourhood of $j_x^k(f) \in \tilde{J}^k(n, m)$ and let $\{x_1, \cdots, x_n\}$ (resp. $\{u_1, \cdots, u_m\}$) be a coordinate system on $\alpha^k(\mathcal{U}^k)$ (resp. $\beta^k(\mathcal{U}^k)$). Then we get a coordinate system $\{x_i (1 \leq i \leq n), u_j (1 \leq j \leq m), p_{j_1 \cdots j_h}^\lambda$ $(1 \leq \lambda \leq m, 1 \leq j_1, \cdots, j_h \leq n, 1 \leq h \leq k+1)\}$ on $\mathcal{U}^{k+1} = (\pi_k^{k+1})^{-1}(\mathcal{U}^k)$ associated with $\{x_1, \cdots, x_n, u_1, \cdots, u_m\}$ introduced by $p_{j_1 \cdots j_h}^\lambda(j_x^{k+1}(f)) = (\partial^h(u_\lambda(f))/\partial x_{j_1} \cdots \partial x_{j_h})(x)$. For a system of differential equations Φ^k defined on $\mathcal{U}^k$ and given by a generator $\{f_1, \cdots, f_r\}$, $p(\Phi^k)$ means the system of differential equations on $\mathcal{U}^{k+1}$ generated by $\{f_i, \partial_j^\# f_l ; i, l=1, \cdots, r$ and $j=1, \cdots, n\}$ where $\partial_j^\# f_l = \partial f_l/\partial x_j + \sum_{\lambda=1}^m p_j^\lambda(\partial f_l/\partial u_\lambda) + \cdots + \sum_{\lambda=1}^m \sum_{j_1 \cdots j_k=1}^n p_{j j_1 \cdots j_k}^\lambda (\partial f_l/\partial p_{j_1 \cdots j_k}^\lambda)$.

Let A or B a system of differential equations defined on $\mathcal{U}^k$ and given by a generator $\{f_1, \cdots, f_r\}$ or $\{g_1, \cdots, g_s\}$, respectively. We denote by $A \supset B$ if the ideal $i(A)$ generated by $\{f_1, \cdots, f_r\}$ in the ring of functions on $\mathcal{U}^k$ contains the ideal $i(B)$ generated by $\{g_1, \cdots, g_s\}$. Denote by $I(A)$ the set of integral points of A.

Theorem 2.1 (Kuranishi [1], p. 142). *Let Φ^l $(l \geq l_0)$ be a system of differential equations on a neighbourhood $\mathcal{U}^l$ of $p^l = j_{x_0}^l(f)$ and assume the following:*

(i) *f is a solution of Φ^l for any $l \geq l_0$.*

(ii) *$\Phi^{l+1} \supset p(\Phi^l)$ on a neighbourhood of $j_{x_0}^l(f)$ for any $l \geq l_0$.*

(iii) *For a suitable open neighbourhood U of p^{l_0}, the triple $(I\Phi^{l_0} \cap U$, $\alpha^{l_0}(U), \alpha^{l_0})$ is a fibred manifold.*

(iv) *The triple $(I\Phi^{l+1} \cap V, I\Phi^l \cap V', \pi_l^{l+1})$ is a fibred manifold for a suitable open neighbourhood V (resp. V') of p^{l+1} (resp. p^l) for any $l \geq l_0$.*

Then there exists an integer l_1 such that Φ^{l+1} and $p(\Phi^l)$ are equal in a neighbourhood of p^{l+1} and such that Φ^l is involutive at p^l for any $l \geq l_1$.

Let Γ be a pseudogroup on $\boldsymbol{R}^m$ which is regular at (x_0, f). In the following proposition, we need not assume that $\dim \mathcal{L}_z < \infty$ where $z = f(x_0)$.

Proposition 2.2. *There exists an integer l_1 such that $P^{l+1} = p(P^l)$ on a neighbourhood of $j_{x_0}^{l+1}(f)$ for any $l \geq l_1$.*

For the proof, we have only to check the conditions (i)–(iv) in Theorem 2.1. For details, refer to [2, p. 468].

Again we assume that $\dim \mathcal{L}_z < \infty$ where $z = f(x_0)$. For the Γ-orbit system P^k at (x_0, f), we set $S^l = \{j_x^l(s)\,;\, s \in \mathcal{S}(P^k),\ x \in \textit{the domain of } s\}$.

Lemma 2.3. *There exists an integer N such that, for any $k \geq N$, we can find a neighbourhood $\mathcal{U}^{k+1}$ (resp. $\mathcal{U}^k$) of $j_{x_0}^{k+1}(f)$ (resp. $j_{x_0}^k(f)$) such that $S^{k+1} \cap \mathcal{U}^{k+1}$ is diffeomorphic to $S^k \cap \mathcal{U}^k$ by the projection π_k^{k+1}.*

Proof. We set $f(k) = \{j_x^k(f)\,;\, x \in \textit{the domain of } f\}$. Since Γ is regular at (x_0, f), by Frobenius theorem for some neighbourhood $\mathcal{U}^k$ of $j_{x_0}^k(f)$ we can get the orbit $\mathcal{J}_p$ of $\mathcal{L}_n^{(k)}$ through $p \in \mathcal{U}^k$ in $\mathcal{U}^k$. Then we have $S^k \cap \mathcal{U}^k = \mathcal{J}^k(f, \mathcal{U}^k) \equiv \bigcup_{p \in f(k) \cap \mathcal{U}^k} \mathcal{J}_p$. Since $\dim \mathcal{L}_z < \infty$, there exists an integer N such that, for any integer $l \geq N$, $\mathcal{J}_{p^{l+1}}$ is a covering space of $\mathcal{J}_{p^l}$ by π_l^{l+1} where $p^l = j_x^l(f)$. Furthermore for a neighbourhood $\tilde{\mathcal{U}}^{k+1}$ (resp. $\tilde{\mathcal{U}}^k$) of p^{k+1} (resp. p^k), $f(k+1) \cap \tilde{\mathcal{U}}^{k+1}$ is diffeomorphic to $f(k) \cap \tilde{\mathcal{U}}^k$ by π_k^{k+1}. Then, for a suitable neighbourhood $\mathcal{U}^{k+1}$ (resp. $\mathcal{U}^k$) of p^{k+1} (resp. p^k), $\mathcal{J}^{k+1}(f, \mathcal{U}^{k+1})$ is diffeomorphic to $\mathcal{J}^k(f, \mathcal{U}^k)$ and therefore $S^{k+1} \cap \mathcal{U}^{k+1}$ is diffeomorphic to $S^k \cap \mathcal{U}^k$.

Lemma 2.4. *For a sufficiently large integer k, if $j_{x_0}^k(s) = j_{x_0}^k(s')$ for two solutions s and $s' : V \to \boldsymbol{R}^m$ of P^k where V is a neighbourhood of x_0, then $s = s'$ on a neighbourhood $\tilde{V} \subset V$ of $x_0 \in \boldsymbol{R}^n$.*

Proof. By Lemma 2.3, there exists an integer N such that, for any $k \geq N$, $S^{k+1} \cap \mathcal{U}^{k+1}$ is diffeomorphic to $S^k \cap \mathcal{U}^k$ by π_k^{k+1}. On the other hand by Proposition 2.2, there exists an integer l_1 such that, if $k \geq l_1$, then $\mathcal{S}(P^{k+1}) = \mathcal{S}(p(P^k))$ in a neighbourhood $\tilde{\mathcal{U}}^{k+1} \subset (\pi_k^{k+1})^{-1}(\mathcal{U}^k)$ of $j_{x_0}^{k+1}(f)$. Therefore, for any solution $s \in \mathcal{S}(P^k) \,|\, \tilde{\tilde{\mathcal{U}}}^{k+1} = \{g \in \mathcal{S}(P^k)\,;\, g(k+1) \subset \tilde{\tilde{\mathcal{U}}}^{k+1}\}$ where $k = \max(N, l_1)$ and $\tilde{\tilde{\mathcal{U}}}^{k+1} = \mathcal{U}^{k+1} \cap \tilde{\mathcal{U}}^{k+1}$, we have $j_{x_0}^{k+1}(s) \in S^{k+1} \cap \tilde{\tilde{\mathcal{U}}}^{k+1}$ and there exist functions $F^\lambda_{j_1 \cdots j_{k+1}} (1 \leq \lambda \leq m,\ 1 \leq j_1, \cdots, j_{k+1} \leq n)]$ on $S^k \cap \pi_k^{k+1}(\tilde{\tilde{\mathcal{U}}}^{k+1})$ such that $p^\lambda_{j_1 \cdots j_{k+1}}(j_x^{k+1}(s)) = F^\lambda_{j_1 \cdots j_{k+1}}(j_x^k(s))$ for $s \in \mathcal{S}(P^k) \,|\, \tilde{\tilde{\mathcal{U}}}^{k+1}$. Inductively $p^\lambda_{j_1 \cdots j_{l+1}}(j_{x_0}^{l+1}(s))$ $(l \geq k+1)$ is determined by $j_{x_0}^k(s)$. If s and s' are in $\mathcal{S}(P^k) \,|\, \tilde{\tilde{\mathcal{U}}}^{k+1}$ and $j_{x_0}^k(s) = j_{x_0}^k(s')$, then $j_{x_0}^l(s) = j_{x_0}^l(s')$ for $l \geq k$. Therefore by the analyticity, we get $s = s'$ on a neighbourhood of x_0.

Now we shall complete the proof of Theorem 1.1. Take any $s \in \mathcal{S}(P^k)$ and assume that s is near to f. Then for a small neighbourhood $\mathcal{U}^k$ of $j_x^k(f)$, we have $j_x^k(s) \in S^k \cap \mathcal{U}^k$. Let us show that $S^k \cap \mathcal{U}^k = I(P^k) \cap \mathcal{U}^k$. Since Γ is regular at (x_0, f), $I(P^k) \cap \mathcal{U}^k$ is a regular submanifold of $\tilde{J}^k(n, m)$ and $\dim I(P^k) \cap \mathcal{U}^k = \dim \tilde{J}^k(n, m) - m_k + n$ because $\{Y_1^k, \cdots, Y_{m_k}^k\}$ is a fundamental system of differential invariants of Γ at $j_{x_0}^k(f)$ and if Y_j^k depends only on

$\{x_1, \cdots, x_n\}$, then the equality $Y_j^k = \lambda_j^k$ holds identically. Since f is a solution of P^k, we get $\mathcal{J}^k(f, \mathcal{U}^k) \subset I(P^k)$. On the other hand, for each $p \in f(k) \cap \mathcal{U}^k$, the orbit $\mathcal{J}_p$ is transversal to $f(k)$. Therefore

$$\dim \mathcal{J}^k(f, \mathcal{U}^k) = n + \dim J^k(n, m) - m_k$$

and we get $\dim \mathcal{J}^k(f, \mathcal{U}^k) = \dim I(P^k) \cap \mathcal{U}^k$. Since $S^k \cap \mathcal{U}^k \supset \mathcal{J}^k(f, \mathcal{U}^k)$ and $S^k \subset I(P^k)$, we get $S^k \cap \mathcal{U}^k = I(P^k) \cap \mathcal{U}^k$ by taking a smaller neighbourhood if necessary. Therefore $S^k \cap \mathcal{U}^k = \mathcal{J}^k(f, \mathcal{U}^k)$. This means that $j_x^k(s) = j_x^k(\sigma \circ f)$ for some $\sigma \in \Gamma$. By Lemma 2.3, $s = \sigma \circ f$ on a neighbourhood of x.

Conversely if $\sigma \in \Gamma$ which is near to the identity, then clearly $\sigma \circ s \in \mathcal{S}(P^k)$ for any $s \in \mathcal{S}(P^k)$. This completes the proof of Theorem 1.1.

References

[1] M. Kuranishi: Lectures on exterior differential systems. Tata Inst. Fund. Res., Bombay (1962).

[2] K. Ueno: Existence and equivalence theorems of automorphic systems. Publ. RIMS, Kyoto Univ., 11, 461–482 (1976).