40. *The Structure of Gravity-Capillary Waves of Permanent Profiles on Deep Water*

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§ 0. **Introduction.** We consider progressive waves of two-dimensional irrotational flow of inviscid incompressible fluid. On this subject, there have been many works of analytical and numerical approach. Recently Okamoto ([9]) pointed out that the proplem is equivariant with respect to the orthogonal group O(2). The O(2)-equivariant bifurcation equations were analyzed by Fujii, Mimura and Nishiura ([4, 5]) and Okamoto ([7, 8]), and they classified the local structures of bifurcating branches at double eigenvalues and secondary branches appearing near them. The numerical results of Chen and Saffman ([1, 2]) are remarkable, but they are insufficient to examine the O(2)-equivariant bifurcation theory. Our purpose is to see how the local structures of the solutions are classified and to investigate the global structures of bifurcation which is hardly clarified by analytical approach. The problem is divided into several cases according as the depth is finite or infinite, and as the gravity and surface tension (capillarity) are taken into account or not. In this paper we will refer to only gravity-capillary waves of infinite depth, leaving other cases to the future work.

§ 1. **Formulation of the problem and numerical method.** We consider progressive waves, which look fixed in a coordinate system moving parallel to the water level with a constant speed c. We use x-y axis and take y-axis upward. Let the free boundary be represented by a function H as $\{(x, y)\,;\, y=H(x)\}$ and we suppose that the wave profile is symmetric about y-axis.

Suppose the flow is two dimensional and irrotational, and the fluid is incompressible and inviscid. Then we can formulate the problem as follows:

Problem. Find functions $H=H(x)$ $(-\infty<x<\infty)$, $U(x, y)$ and $V(x, y)$ $(-\infty<x<\infty,\ -\infty<y<H(x))$ satisfying the followings:

(1) U and V are harmonic in $\{-\infty<x<\infty,\ -\infty<y<H(x)\}$ and $w=w(z)=U+iV$ is a complex analytic function of $z=x+iy$,

(2) $H(x)$ and $\dfrac{dw}{dz}$ are periodic functions of x with a period L,

(3) $V=0$ on $y=H(x)$,

(4) $V=-\infty$ on $y=-\infty$, $\dfrac{\partial U}{\partial x}\longrightarrow c$ as $y\longrightarrow-\infty$,

(5) $$\frac{1}{2}(U_x^2+V_x^2)+gH-T\left(\frac{H_x}{(1+H_x^2)^{1/2}}\right)_x=\text{constant} \qquad \text{on } y=H(x),$$

where c, g and T are positive constants. Subscripts mean differentiations. g is the gravity acceleration and T is the surface tension coefficient.

By making use of the Stokes expansion, we seek a solution of the following form:

(6) $$z=x+iy=\frac{w}{c}+\frac{iL}{2\pi}\sum_{j=1}^{\infty}\frac{C_j}{j}\exp\left(-\frac{2j\pi iw}{cL}\right)+\frac{iLC_0}{2\pi},$$

where $C_j=A_j+iB_j$ (A_j, B_j: real), $i=\sqrt{-1}$ (see [1, 2]). $\{V\le 0\}$ is a fluid region. From (3) and the assumption on symmetry, the free surface $\{(x,y): y=H(x)\}$ is given by putting $B_j=0$ and $V=0$ in (6):

$$x=\frac{U}{c}+\frac{L}{2\pi}\sum_1^{\infty}\frac{A_j}{j}\sin\frac{2j\pi U}{cL}, \qquad y=\frac{L}{2\pi}\sum_1^{\infty}\frac{A_j}{j}\cos\frac{2j\pi U}{cL}+\frac{LA_0}{2\pi}.$$

In these (x, y), put $\xi=2\pi U/cL$ and take A_0 such that $\xi=0$ corresponds to $(x, y)=(0, 0)$, then (5) is rewritten as:

(7) $$\frac{\mu}{2}\frac{1}{X'^2+Y'^2}+Y-\kappa\frac{X'Y''-X''Y'}{(X'^2+Y'^2)^{3/2}}=\text{constant},$$

where $'$ means differentiations about ξ,

(8) $$\mu=\frac{2\pi c^2}{gL}, \qquad \kappa=\frac{4\pi^2 T}{gL^2},$$

(9) $$X(\xi)=\xi+\sum_1^{\infty}\frac{A_j}{j}\sin j\xi, \qquad Y(\xi)=\sum_1^{\infty}\frac{A_j}{j}(\cos j\xi-1).$$

Chen and Saffman ([1, 2]) used the above equation (7), however we prefer the following differential form (7′) of (7) since it is convenient in order to apply bifurcation theory given by Crandall and Rabinowitz ([3]) and others:

(7′) $$\frac{\mu}{2}\frac{\partial}{\partial\xi}\frac{1}{X'^2+Y'^2}+Y'-\kappa\frac{\partial}{\partial\xi}\frac{X'Y''-X''Y'}{(X'^2+Y'^2)^{3/2}}=0.$$

The task now is to solve (7′) with (8)–(9), i.e. to find μ, κ and A_j, $j=1, 2, \cdots$ satisfying (7′)(8)(9).

Let us rewrite these in an abstract operator form as follows: Put $A=(A_1, A_2, \cdots)$. Then define quantities F and G by

(10) $$F(\kappa,\mu,A)(\xi)=\frac{\mu}{2}\frac{1}{X'^2+Y'^2}+Y-\kappa\frac{X'Y''-X''Y'}{(X'^2+Y'^2)^{3/2}},$$

(11) $$G(\kappa,\mu,A)(\xi)=\frac{\partial}{\partial\xi}F(\kappa,\mu,A)(\xi).$$

Then our problem is to solve the following nonlinear equation:

(12) $$G(\kappa,\mu,A)(\xi)\equiv 0 \qquad (\xi\in[0,2\pi)).$$

Here we see that

i) $G(\kappa, \mu, 0)\equiv 0$, i.e. $A=(0, 0, \cdots)$ is a trivial solution for all κ and μ,

ii) a Fréchet derivative of F and G at $A=(0, 0, \cdots)$ fails to be an isomorphism if and only if

(13) $$\mu=m^{-1}+m\kappa \qquad (m=1, 2, \cdots).$$

Now let κ be fixed and μ_m be taken so as to satisfy (13) for some integer

m. (κ, μ_m) is called a bifurcation point of the mth primary bifurcation and we can solve this problem by the well known method for bifurcation from simple eigenvalue ([6]). Solving (12) is equivalent to equating all the Fourier coefficients of G to zero, and here we may consider only the coefficients of $\sin j\xi$. For it is easily seen from (9) that $F(\xi)$ is even and $G(\xi)$ is odd function of ξ.

By the truncation we naturally have the following discrete problem:

$$(14)\quad \begin{aligned} H_j(\lambda, A_1, \cdots, A_n; \kappa) &\equiv \int_0^{2\pi} G(\kappa, \mu_m+\lambda, A_1, \cdots, A_n, 0, 0, \cdots)(\xi)\sin j\xi d\xi \\ &= -j\int_0^{2\pi} F(\kappa, \mu_m+\lambda, A_1, \cdots, A_n, 0, 0, \cdots)(\xi)\cos j\xi d\xi \\ &= 0 \quad (1\le j\le n). \end{aligned}$$

The remaining equation is taken to control bifurcation parameter, for example, we use

$$(15\text{-}1)\qquad H_{n+1}(\lambda, A_1, \cdots, A_n; \kappa)\equiv \lambda-\tilde{\lambda}=0, \quad \text{or}$$

$$(15\text{-}2)\qquad H_{n+1}(\lambda, A_1, \cdots, A_n; \kappa)\equiv A_j-\tilde{A}_j=0,$$

where $\tilde{\lambda}$ and $\tilde{A}_j$ are suitably given constants.

Then $H\equiv(H_1, H_2, \cdots, H_{n+1})^t=0$ gives $n+1$ nonlinear equations for the $n+1$ unknowns λ, A_j $(1\le j\le n)$, and this system can be solved by Euler-Newton's method:

$$DH^p\cdot\begin{bmatrix} \lambda^{p+1}-\lambda^p \\ A_1^{p+1}-A_1^p \\ \vdots \\ A_n^{p+1}-A_n^p \end{bmatrix} = -H(\lambda^p, A_1^p, \cdots, A_n^p).$$

Here DH^p is the Jacobian matrix of H at $(\lambda, A_1, \cdots, A_n)=(\lambda^p, A_1^p, \cdots, A_n^p)$, and the FFT method is effectively used to compute this matrix. In fact, the kth row of DH is composed of cosine coefficients such as

$$\begin{cases} \dfrac{\partial H_j}{\partial\lambda} = -j\displaystyle\int_0^{2\pi}\frac{\partial F}{\partial\lambda}\cos j\xi d\xi \quad (k=1), \\ \dfrac{\partial H_j}{\partial A_{k-1}} = -j\displaystyle\int_0^{2\pi}\frac{\partial F}{\partial A_{k-1}}\cos j\xi d\xi \quad (k\ne 1), \end{cases}$$

where $\partial F/\partial\lambda$ and $\partial F/\partial A_{k-1}$ can be written concretely.

Next, we will observe combination waves which have two different wave numbers. In (13) we can see there are the cases that $\mu_{m_1}=\mu_{m_2}$ for $m_1\ne m_2$. Namely, if $\kappa=1/m_1m_2$ $(m_1\ne m_2)$ then the m_1th primary bifurcation point coincides with the m_2th one. This causes the existence of (m_1, m_2) combination waves. Numericaly, we can solve in the same way as above, by taking suitable initial values in Euler-Newton's iteration. When κ is close to $1/m_1m_2$, there appears a secondary bifurcation which we can find by checking a change of sign of the determinant of the Jacobian matrix. To study bifurcation from double eigenvalues and secondary bifurcations, we can follow the procedures of [4, 5] and [7]. Here we remark that there is no bifurcation point of multiplicity ≥ 3. (See [9].)

§ 2. **Numerical results.** As described in § 1, if κ is close to 0.5 then

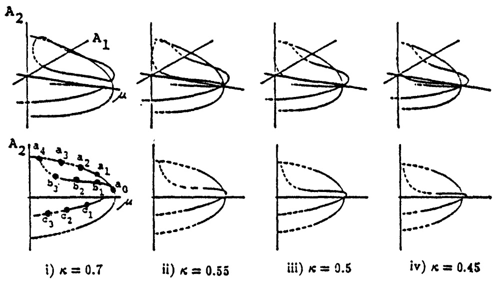

Fig. 1

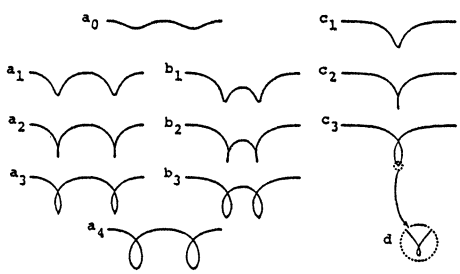

Fig. 2

there are combination waves of mode 1 and mode 2. Figure 1 shows the bifurcation diagram of the case $\kappa>0.5$, $\kappa=0.5$ and $\kappa<0.5$. When $\kappa<0.7$, there is a turning point (limit point). This fact seems to be new. Topologically they agree with one type of O(2)-equivariant bifurcation structures given in [5, 7]. In Fig. 2 we given an example of wave configurations on bifurcation branch of the case $\kappa=0.7$. The bifurcating solutions on the dotted lines are meaningless from physical viewpoint, since they have self-intersection. But mathematically, we can regard them as our solutions. We think the inclusion of such solutions is convenient in order to make the present bifurcation phenomenon clearer. For instance, the secondary branch in Fig. 1 will meet the primary one again, which agree with [5] and [7]. It must be important to consider the other cases of finite depth in the same way. We will report them elsewhere.

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