28. On a Canonical Standard Form of Second Order Linear Ordinary Differential Equations with a Small Parameter

By Atsushi YOSHIKAWA*)
Department of Applied Science, Kyushu University

(Communicated by Kôsaku YOSIDA, M. J. A., April 13, 1987)

1. **Introduction.** In many standard texts of singular perturbation of ordinary differential equations, the function $v(r,\varepsilon)=a+b-be^{-r/\varepsilon}$, $r>0$, $\varepsilon>0$, a, b being constants, serves as a most basic illustration of the initial layer phenomena at $r=0$ as $\varepsilon\downarrow 0$ (see, e.g., [1], [3], [5], [6]). The function v satisfies the equation

$$\varepsilon d^2v/dr^2+dv/dr=0, \qquad r>0,\ \varepsilon>0, \tag{1}$$

with the initial data $v(0,\varepsilon)=a$, $\varepsilon dv(0,\varepsilon)/dr=b$.

Now consider a slightly more general initial value problem:

$$\varepsilon^2d^2u/dt^2+\varepsilon p(t,\varepsilon)du/dt+q(t,\varepsilon)u=0, \qquad t>0,\ \varepsilon>0, \tag{2}$$

with the initial data $u(0,\varepsilon)=f^0(\varepsilon)$, $\varepsilon du(0,\varepsilon)/dt=f^1(\varepsilon)$.

The purpose of the present note is to show that the problem (2) can be reduced to the problem (1) by a change of the dependent and independent variables which incorporates the initial layer at $t=0$ as $\varepsilon\downarrow 0$ provided the coefficients $p(t,\varepsilon)$ and $q(t,\varepsilon)$ are taken from the appropriate asymptotic class, $p(t,\varepsilon)$ positive and $q(t,\varepsilon)$ small in the sense to be specified below. The asymptotic class will be given shortly and will also be shown to contain the solution $u(t,\varepsilon)$ itself of (2) as well as the transformed dependent and independent variables when expressed in the arguments t,ε. These will show relevance of the asymptotic class in the present context.

2. **The asymptotic class.** Let $\sigma\in\boldsymbol{R}$. We denote by $\mathcal{A}^\sigma$ the set of C^∞ functions $f(t,\varepsilon)$, $t\geqq 0$, $\varepsilon>0$, with the estimate

$$|\partial_t^i\partial_\varepsilon^j f(t,\varepsilon)|\leqq C_{i,j,t_0,\varepsilon_0}\varepsilon^{\sigma-i-j}$$

when $0\leqq t\leqq t_0$, $0<\varepsilon\leqq\varepsilon_0$, for any choice of non-negative integers i, j and positive numbers t_0, ε_0. A typical example of the elements of the class $\mathcal{A}^\sigma$ is given by

$$\varepsilon^\sigma(h(t)+H(t/\varepsilon)), \qquad t\geqq 0,\ \varepsilon>0, \tag{3}$$

where $h(t)$ is a C^∞ function of $t\geqq 0$, or $h\in\mathcal{E}_+$, and $H(s)$ is C^∞ and rapidly decreasing as $s\to+\infty$, or $H\in\mathcal{S}_+$ in short. Namely, $H(s)$ satisfies the estimate $\sup_{s\geqq 0}|s^i\partial_s^jH(s)|<+\infty$ for any $i, j=0, 1, \cdots$. h and H in (3) are uniquely determined by the values of the sum (3). The totality of the elements of the form (3) will be denoted by Γ^σ.

Our asymptotic class $\mathcal{A}_d^0$ is then given as follows. $h(t,\varepsilon)\in\mathcal{A}_d^0$ if there are sequences of functions $h_j(t)\in\mathcal{E}_+$ and $H_j(s)\in\mathcal{S}_+$, $j=0, 1, \cdots$, such that

*) Supported in part by Grant-in-Aid for Scientific Research, Ministry of Education, Science and Culture, Japan, No. 61540118.

for any $N=1,2,\cdots$,

$$h(t,\varepsilon)-\sum_{j=0}^{N}\varepsilon^{j}(h_j(t)+H_j(t/\varepsilon))\in\mathcal{A}^{N+1}. \tag{4}$$

The relation (4) will be written as $h(t,\varepsilon)\sim\sum\varepsilon^{j}(h_j(t)+H_j(t/\varepsilon))$ in short. The formal sum is uniquely determined and is called the asymptotic expansion of $h(t,\varepsilon)$. It is not difficult to see $\mathcal{A}_d^0\subset\mathcal{A}^0$ and that the class $\mathcal{A}^{\sigma}$ is closely related to Hörmander's symbol class $S_{1,1}^{-\sigma}$ (see [2]). In particular, given a formal sum, one can always construct an element in $\mathcal{A}_d^0$ so that (4) holds good. The class $\mathcal{A}_d^0$ is a Fréchet algebra as well as $\mathcal{A}^{\infty}=\bigcap_{\sigma}\mathcal{A}^{\sigma}$ and the direct sum $\mathcal{D}^0=\sum_{i=0}^{\infty}\Gamma^{i}$. By means of the Fréchet algebra exact sequence $0\to\mathcal{A}^{\infty}\to\mathcal{A}_d^0\to\mathcal{D}^0\to 0$, one can carry calculi in $\mathcal{A}_d^0$ to those in $\mathcal{D}^0$. Details of these formal properties will be discussed elsewhere. However, for later convenience, we here note that if $\varphi(x)$ is a C^{∞} function of $x\in\boldsymbol{R}$ and $h(t,\varepsilon)\in\mathcal{A}_d^0$ is real valued, then the composite function $\varphi(h(t,\varepsilon))\in\mathcal{A}_d^0$.

3. Main result. Now we return to Equation (2). We take

$$p(t,\varepsilon),\qquad q(t,\varepsilon)\in\mathcal{A}_d^0. \tag{5}$$

Then $p(t,\varepsilon)\sim\sum\varepsilon^{i}(p_i(t)+P_i(t/\varepsilon))$, $q(t,\varepsilon)\sim\sum\varepsilon^{j}(q_j(t)+Q_j(t/\varepsilon))$.

Our positive-small requirements are:

$$p_0(t)>0,\qquad t\geqq 0, \tag{6}$$

$$q_0(t)\equiv 0,\qquad t\geqq 0, \tag{7}$$

and, for any $T>0$, there is $\delta>0$ such that

$$\int_0^T(p_0(t)+P_0(t/\varepsilon))u'(t)^2dt+\frac{1}{\varepsilon}\int_0^T Q_0(t/\varepsilon)u(t)u'(t)\,dt\geqq\delta\int_0^T u'(t)^2dt \tag{8}$$

for all $u\in\mathcal{E}_+$ with $u(0)=0$ and $\varepsilon>0$. Here $u'(t)=du(t)/dt$.

Remark. For such u, if normalized as

$$\int_0^T u'(t)^2dt=1,$$

$$\left|\frac{2}{\varepsilon}\int_0^T Q_0(t/\varepsilon)u(t)u'(t)\,dt\right|\leqq\sup_{s\geqq 0}|sQ_0(s)|+\int_0^{\infty}|Q_0'(s)|\,sds.$$

If $Q_0(s)\geqq 0\geqq Q_0'(s)$, then

$$\int_0^T Q_0(t/\varepsilon)u(t)u'(t)dt\geqq 0.$$

We also require that, for $\varepsilon>0$ and $t\geqq 0$,

$$p(0,\varepsilon)>0, \tag{9}$$

$$a(t,\varepsilon)=\frac{1}{4}p(t,\varepsilon)^2+\frac{1}{2}\varepsilon\partial_t p(t,\varepsilon)-q(t,\varepsilon)>0. \tag{10}$$

Remark. If $u(t,\varepsilon)$ is a solution of Equation (2), then

$$-\varepsilon^2d^2w/dt^2+a(t,\varepsilon)w=0 \tag{11}$$

holds for

$$w(t,\varepsilon)=u(t,\varepsilon)\exp\left\{\frac{1}{2\varepsilon}\int_0^t p(r,\varepsilon)dr\right\}.$$

Lemma 1. *Assume* (6). *For any* $F(s)\in\mathcal{S}_+$ *and* $A,B\in\boldsymbol{C}$, *there are uniquely determined* $U(s)\in\mathcal{S}_+$ *and* $c\in\boldsymbol{C}$ *which satisfy*

$$d^2U/ds^2+(p_0(0)+P_0(s))dU/ds+Q_0(s)U=F(s)-Q_0(s)c,$$

$$U(0)=A-c,\qquad dU(0)/ds=B.$$

In fact, Banach's closed range theorem is applicable. c is determined from

$$c\left\{\int_0^\infty Q_0(s)G(s)ds+G(0)\right\}=\int_0^\infty F(s)G(s)ds+AG(0)$$
$$+B\{(p_0(0)+P_0(0))G(0)-G'(0)\},$$

where $G(s)$ is slowly increasing as $s\to+\infty$ satisfying

$$d^2G/ds^2-(p_0(0)+P_0(s))dG/ds+(Q_0(s)-P_0'(s))G=0.$$

$G(s)$ is unique up to a constant multiple. It can be shown

$$\int_0^\infty Q_0(s)G(s)ds\neq -G(0)$$

unless $G(s)=0$.

Lemma 2. *Assume* (5), (6), (7) *and* (8). *Let* $u^i(t,\varepsilon)$, $i=0,1$, *be the solutions of Equation* (2) *with the initial data* $u^i(0,\varepsilon)=\delta_{i0}$, $\varepsilon\partial_t u^i(0,\varepsilon)=1$, $i=0,1$. *Then* $u^i(t,\varepsilon)\in\mathcal{A}_d^0$. *Furthermore, under the additional requirements* (9) *and* (10), *we have*

$$u^0(t,\varepsilon)>u^1(t,\varepsilon)>0, \qquad t>0,\ \varepsilon>0.$$

Moreover, $1/u^0(t,\varepsilon)\in\mathcal{A}_d^0$.

Proof. For the first half, transfer Equation (2) into the space $\mathcal{D}^0$ of the formal sums. Then (6), (7) and Lemma 1 yield to a formal solution (cf. [5]). That this leads to the solutions of (2) by modifying additional terms in $\mathcal{A}^\infty$ can be shown by a kind of the energy estimate using (8) (cf. [1]). For the second half, use (11). Here (10) is essential. (9) is used to ensure $u^0>u^1$. Consult, e.g., [4].

Now our main result is the following

Theorem. *Assume* (5), (6), (7), (8), (9) *and* (10). *Let* $u(t,\varepsilon)$ *be the solution of Equation* (2) *with the initial data* $\varepsilon^j\partial_t^j u(0,\varepsilon)=f^j(\varepsilon)\in\mathcal{A}_d^0$, $j=0,1$. *Let*

$$r(t,\varepsilon)=-\varepsilon\log\{1-u^1(t,\varepsilon)/u^0(t,\varepsilon)\}\in\mathcal{A}_d^0$$

and

$$v(r(t,\varepsilon),\varepsilon)=u(t,\varepsilon)/u^0(t,\varepsilon)\in\mathcal{A}_d^0.$$

Then $v(r,\varepsilon)$ *satisfies Equation* (1) *so that*

$$v(r,\varepsilon)=f^1(\varepsilon)+(f^0(\varepsilon)-f^1(\varepsilon))e^{-r/\varepsilon}.$$

The verification is straightforward. Observe that $r(0,\varepsilon)=0$, $\partial_t r(t,\varepsilon)>0$, $t\geqq 0$, $\varepsilon>0$, with $\partial_t r(0,\varepsilon)=1$. Note that even $\varepsilon^{-1}r(t,\varepsilon)\in\mathcal{A}_d^0$ holds.

Remark. $R(t,\varepsilon)=(\partial_t r(t,\varepsilon))^{-1/2}\in\mathcal{A}_d^0$ satisfies

$$-\varepsilon^2d^2R/dt^2+a(t,\varepsilon)R=\frac{1}{4}R^{-3}$$

with $R(0,\varepsilon)=1$. Here $a(t,\varepsilon)\in\mathcal{A}_d^0$ given by (10).

References

[1] Eckhaus, W.: Asymptotic Analysis of Singular Perturbations. North-Holland, Amsterdam (1979).

[2] Hörmander, L.: Pseudo-differential operators and hypoelliptic equations. Amer. Math. Soc. Symp. on Singular Integrals, 138–183 (1966).

[3] Kevorkian, J., and Cole, J. D.: Perturbation Methods in Applied Mathematics. Springer-V., New York (1981).
[4] Miyatake, S.: Constructions of eigenfunctions for the Sturm-Liouville operator by comparison method. J. Math. Kyoto Univ., **23**, 1–11 (1983).
[5] Vasil'eva, A. B. and Butuzov, V. F.: Asymptotic Expansions of the Solutions of Singularly Perturbed Equations. Nauka, Moscow (1973) (in Russian).
[6] Wasow, W.: Asymptotic Expansions for Ordinary Differential Equations. Interscience, New York (1965).