27. On One Fixed Point Actions on Spheres

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1. Introduction. In this paper G will denote a finite group, and $S=S^k$ the standard oriented sphere of dimension k. Unless specified to the contrary, group actions on smooth manifolds mean smooth group actions. The G-actions on manifolds with exactly one G-fixed point are called *one fixed point actions*. Suppose that (G, S^k), $k \geqq 5$, is a one fixed point action with fixed point x. Take a closed G-equivariant disk neighborhood N of x in S. Then the disk $D=S-\mathrm{Int}\,N$ has the fixed point free induced G-action. Here a fixed point free G-action means an action without G-fixed points. The class of finite groups having fixed point free actions on disks is restrictive. For example, by Smith's theorem the groups of prime power order do not belong to the class ([11]). J. Greever showed in [3] that if $|G|<60$ and $|G| \neq 36$, then G does not lie in the class. In 1974, R. Oliver [7] completely decided the class: for example, the least order of groups in the class is 60. It is the order of A_5, the alternating group on five letters. On the other hand, a sort of existence theorem of fixed point free actions on disks was given first by E. Floyd-R. Richardson in 1958 ([1]). That is, they proved that A_5 has a fixed point free simplicial action on a disk D. Since we have a one fixed point topological action on the sphere $S=D/\partial D$, we may have one fixed point smooth actions on spheres. In fact, E. Stein [12] succeeded in 1976 to show that $SL(2,5)$, the binary icosahedral group, has one fixed point actions on S^7. In 1978, T. Petrie asserted in [8] that (1) $SL(2,F)$, $PSL(2,F)$ with characteristic of F odd and (2) the odd order abelian groups having at least three non-cyclic Sylow subgroups have one fixed point actions on homotopy spheres: These homotopy spheres are likely to be the standard spheres from the construction. (For further information see [9] and [10].) Thus there are various one fixed point actions on spheres.

We are led to ask what is the least dimension of the standard spheres with one fixed point actions of finite groups. We denote the dimension by LD in this paper. It is easy to see that LD is neither 1 nor 2.

2. Results. Our first result is that $LD \leqq 6$: More precisely,

Theorem A ([5]). *There exist one fixed point actions of A_5 on S^6.*

Outline of the proof. On the analogy of [9] and [10], we obtain a G-framed normal map $f: X \to Y = S(\boldsymbol{R} \oplus V)$ with $|X^G|=1$, $G=A_5$, and H-framed normal cobordisms $F_H: W_H \to Y \times I$ between f and $id: X \to X$ for all the proper subgroups H of G. Here V is a real G-module of dimension 6 which satisfies that (1) $\dim V^H=0$ for H dihedral and (2) $\dim V^C=2$ for $C\neq 1$ cyclic. By equivariant surgery of f and F_H, we can arrange them so that (3) f^H is a homotopy equivalence for H with $1\neq H\neq G$, (4) ${F_H}^K$ is a homotopy equivalence for K dihedral, $K\subset H$, and (5) f is 3-connected. We obtain the G-surgery obstruction $\sigma(f)\in L_6(Z[G])$, the Wall group of homotopy equivalence, to converting f to a homotopy equivalence. It is well known that $\sigma(f)=0$ if $\mathrm{res}_H\sigma(f)=0$ in $L_6(Z[H])$ for all the hyperelementary subgroups H of G. It is sufficient to show $\mathrm{res}_H\sigma(f)=\sigma(\mathrm{res}_H f)=0$. By Petrie's surgery theory, $\sigma(\mathrm{res}_H f)=0$ if we can make F_H so that ${F_H}^P$ is a mod p homology equivalence for any non-trivial p-subgroup P of H, p prime. Since $\dim {W_H}^P=1$ or 3, the proof is related to $N_H(P)/P$-surgery theory of 3-dimensional manifolds. The conclusion follows from a vanishing lemma of the $N_H(P)/P$-surgery obstruction group with coefficient ring $Z_{(p)}$.

A word about the motivation of Theorem A is in order. E. Laitinen-P. Traczyk [4] proved that if a homotopy sphere Σ^k with $k\geqq 5$ has a one fixed point G-action such that $\dim\Sigma^g\leqq 2$ for any element $g\neq 1$ of G, then $\Sigma=S^6$ and $G=A_5$. Our Theorem A gives the converse.

In the rest of this paper, Σ stands for a *four dimensional oriented homotopy sphere* and Ξ for a *four dimensional homology sphere of integer coefficients*. We denote by Ξ^G_h the totality of h-dimensional connected components of the G-fixed point set Ξ^G.

Theorem B ([6]). *For any G-action on Ξ^4, one has $|\Xi^G_0|\leqq 2$.*

Outline of the proof. Suppose that the G-action is effective and $|\Xi^G_0|\geqq 3$. Let K be the subgroup consisting of elements of G preserving an orientation of Ξ. We can show that K is not solvable nor isomorphic to A_5, and furthermore that each Sylow subgroup of K is cyclic or dihedral. By using Suzuki's theorem [13], we see that K has a subgroup H such that $[K:H]\leqq 2$, $H=Z\times A$, where Z is a solvable group and $A\cong A_5$. Since $|\Xi^A|\geqq 3$, we have $\Xi^A\cong S^1$. From the relation $\Xi^G=(((\Xi^A)^H)^K)^G$, we conclude that Ξ^G is a sphere. This contradicts the assumption $|\Xi^G_0|\geqq 3$.

A striking theorem has been recently announced by M. Furuta.

Theorem (M. Furuta [2]). *Any finite group G can not act on Σ^4 in such a way that* (1) $|\Sigma^G|=1$ *and* (2) *each element of G preserves the orientation of Σ.*

If we assume his theorem, then we have:

Theorem C ([6]). *Provided $\Sigma^G_0\neq\emptyset$, then Σ^G consists of exactly two points.*

Outline of the proof. Suppose that the G-action is effective and $|\Sigma^G_0|=1$. Let K be as in the proof of Theorem B. We can show that K is not solvable, hence $\dim\Sigma^K\leqq 1$. It holds that $\chi(\Sigma^K_0)=\chi(\Sigma^K)\equiv\chi(\Sigma^G)=\chi(\Sigma^G_0)=1$ (mod 2). Hence $|\Sigma^K_0|$ is an odd number. If $\Sigma^K_1\neq\emptyset$, then $K=A_5$ and furthermore $\Sigma^K\cong S^1$. In this case $|\Sigma^G_0|=1$ does not occur. Thus we have $\Sigma^K=\Sigma^K_0$. By Furuta's theorem we have $|\Sigma^K|\neq 1$, hence $|\Sigma^K_0|\geqq 3$. This contradicts Theorem B.

It follows that LD is 3, 5 or 6.

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References

[1] E. Floyd and R. Richardson: An action of a finite group on an n-cell without stationary points. Bull. Amer. Math. Soc., **65**, 73–76 (1959).

[2] M. Furuta: A remark on fixed points of finite group actions on S^4 (in preparation).

[3] J. Greever: Stationary points for finite transformation groups. Duke Math. J., **27**, 163–170 (1960).

[4] E. Laitinen and P. Traczyk: Pseudofree representations and 2-pseudofree actions on spheres. Proc. Amer. Math. Soc., **97**, 151–157 (1986).

[5] M. Morimoto: S^6 has one fixed point actions of A_5 (preprint).

[6] ——: Four dimensional homotopy spheres do not have one fixed point actions (in preparation).

[7] R. Oliver: Fixed-point sets of group actions on finite acyclic complexes. Comment. Math. Helv., **50**, 155–177 (1975).

[8] T. Petrie: Pseudoequivalences of G-manifolds. Proc. Symp. in Pure Math. (AMS), **32**, 169–210 (1978).

[9] ——: One fixed point actions on spheres, I. Adv. Math., **46**, 3–14 (1982).

[10] ——: One fixed point actions on spheres, II. ibid., **46**, 15–70 (1982).

[11] P. Smith: Fixed-points of periodic transformations. Amer. Math. Soc. Coll. Pub., **27**, 250–373 (1942).

[12] E. Stein: Surgery on products with finite fundamental group. Topology, **16**, 473–493 (1977).

[13] M. Suzuki: On finite groups with cyclic Sylow subgroups for all odd primes. Amer. J. Math., **77**, 657–691 (1955).