21. On Siegel Series for Hermitian Forms

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(Communicated by Shokichi IYANAGA, M. J. A., March 12, 1987)

Let K be an imaginary quadratic number field of discriminant d_K with ring of integers o_K. We let $\Omega_n(K)$ denote the set of hermitian matrices in $M_n(K)$ and put $\Omega_n(o_K)=\Omega_n(K)\cap M_n(o_K)$. An element $H=(h_{ij})\in\Omega_n(K)$ is called *semi-integral* if $h_{kk}\in Z$ and $\sqrt{d_K}\,h_{ij}\in o_K$ $(i\neq j)$. Denote by $\Lambda_n(K)$ the set of semi-integral matrices in $\Omega_n(K)$. For an element H in $\Lambda_n(K)$, we define a *singular series* by

$$b(s, H)=\sum_R \nu(R)^{-s}\exp[2\pi i tr(HR)], \qquad s\in \boldsymbol{C},$$

where R runs over all hermitian matrices mod $\Omega_n(o_K)$ and $\nu(R)$ denotes the determinant of the denominator of R (cf. [1]). If $\mathrm{Re}(s)>2n$, then the series is absolutely convergent. In the case of quadratic forms, this series was studied by Siegel [6], Kaufhold [2], Shimura [5] and Kitaoka [3]. The purpose of this note is to give an explicit formula for the series $b(s, H)$ under a certain condition.

In the rest of this note, we assume that the class number of K is 1 and $n=2$. For each hermitian matrix R in $M_2(K)$, we have a unique decomposition $R\equiv\sum R_p$ mod $\Omega_2(o_K)$ where R_p is a hermitian matrix in $M_2(K)$ such that $\nu(R_p)$ is a power of rational prime p. Therefore we have a decomposition

$$b(s, H)=\prod_p b_p(s, H),$$
$$b_p(s, H)=\sum_{R_p}\nu(R_p)^{-s}\exp[2\pi i tr(HR_p)],$$

where R_p runs over all hermitian matrices mod $\Omega_2(o_K)$ such that $\nu(R_p)$ is a power of rational prime p. Thus our problem is reduced to finding a formula for $b_p(s, H)$. The series $b_p(s, H)$ was studied by Shimura in [5] under the general situation and is called *Siegel series associated with H*.

We fix a rational prime p. For each non-zero matrix H in $\Lambda_2(K)$, and put $d_1(H)=\max\{m\in Z \mid m^{-1}H\in\Lambda_2(K)\}$ and $p^{\alpha(H)}\|d_1(H)$. When H is non-singular we determine the integers $a(H)$, $d(H)$ and $d_p(H)$ by $p^{a(H)}\|d(H)=|\sqrt{d_K}\,H|$ (the determinant of $\sqrt{d_K}\,H$), $d(H)=p^{a(H)}d_p(H)$. We note that $a(H)\geqq 2\alpha(H)\geqq 0$.

The first result can be stated as follows.

Theorem 1. *Let H be a non-zero matrix in $\Lambda_2(K)$ and $\chi(\cdot)$ denote the Kronecker symbol of K.*

(1) *If $|H|\neq 0$, then*

$$b_p(s, H)=(1-p^{-s})(1-\chi(p)p^{1-s})F_p(s, H),$$

where

$$F_p(s,H)=\begin{cases}\sum_{l=0}^{\alpha} p^{l(3-s)}\{\sum_{m=0}^{[a/2]-l} p^{m(4-2s)}+\chi(p)p^{2-s}\sum_{m=0}^{[(a-1)/2]-l} p^{m(4-2s)}\} & \\ & \text{if } \chi(p)\neq 0,\\ \sum_{l=0}^{\alpha} p^{l(3-s)}\{1+\chi(d_p)(1-\chi^2(d/p^{2l}))p^{(a-2l)(2-s)}\} & \text{if } \chi(p)=0.\end{cases}$$

(2) *If* $|H|=0$, *then*

$$b_p(s,H)=(1-p^{-s})(1-\chi(p)p^{1-s})(1-\chi(p)p^{2-s})^{-1}F_p(s,H),$$

where

$$F_p(s,H)=\sum_{l=0}^{\alpha} p^{l(3-s)}.$$

Here $[x]$ *is the largest integer* $\leqq x$. *For simplicity we put* $\alpha=\alpha(H)$, $a=a(H)$, $d=d(H)$ *and* $d_p=d_p(H)$.

Remark. When $H=0^{(2)}$ (the zero matrix of degree 2) we have

$$b_p(s,0^{(2)})=(1-p^{-s})(1-\chi(p)p^{1-s})(1-\chi(p)p^{2-s})^{-1}(1-p^{3-s})^{-1}$$

for any prime p. The general formula for $b_p(s,0^{(n)})$ has been obtained in [5].

Corollary. *Let* H *and* $\chi(\cdot)$ *be as in Theorem. If we put*

$$F(s,H)=\begin{cases}\prod_{p\mid d} F_p(s,H) & \text{if } |H|\neq 0\\ \prod_{p\mid d_1} F_p(s,H) & \text{if } |H|=0,\end{cases}$$

then we have

$$b(s,H)=\begin{cases}\zeta(s)^{-1}L(s-1,\chi)^{-1}F(s,H) & \text{if } |H|\neq 0\\ \zeta(s)^{-1}L(s-1,\chi)^{-1}L(s-2,\chi)F(s,H) & \text{if } |H|=0,\end{cases}$$

where $\zeta(s)$ *is the Riemann zeta function and* $L(s,\chi)$ *is the Dirichlet L-function attached to* χ *and* $d_1=d_1(H)$. *Furthermore* $F(s,H)$ *can be continued as a holomorphic function in s to the whole C and satisfies*

$$F(s,H)=\begin{cases}\varepsilon(H)|d|^{2-s}F(4-s,H) & \text{if } |H|\neq 0\\ d_1^{3-s}F(6-s,H) & \text{if } |H|=0,\end{cases}$$

where $\varepsilon(H)=sgn(d)=sgn(-|H|)$.

Now we denote by H_n the hermitian upper-half space of degree n. For each Z in H_n, we put $I(Z)=(2i)^{-1}(Z-{}^t\bar{Z})$. Then $I(Z)$ is a positive hermitian matrix. Following to Kaufhold [2], we consider a Dirichlet series $\phi^{(2)}(Z,s)$ corresponding to the hermitian modular group of degree 2 defined by

$$\phi^{(2)}(Z,s)=|I(Z)|^{s/2}\sum_{\{C,D\}}\|CZ+D\|^{-s},\qquad (Z,s)\in H_2\times C.$$

Generalized hypergeometric functions have been studied by Shimura in [4]. If we combine his results and the above corollary, we obtain the following theorem.

Theorem 2. *We define*

$$\rho(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s),\ \rho_\chi(s)=|d_K|^{s/2}\pi^{-s/2}\Gamma((s+1)/2)L(s,\chi),$$

where $\Gamma(s)$ *is the ordinary gamma function. If we put*

$$\xi(s)=\rho(s)\rho_\chi(s-1)\phi^{(2)}(Z,s),$$

then ξ *can be continued as a meromorphic function in s to the whole C and satisfies*

$$\xi(s)=\xi(4-s).$$

Remark. From Theorem 1 we can derive a formula for Fourier coefficients of holomorphic Eisenstein series for the hermitian modular group of degree 2.

References

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