20. The Moduli Space of Hermite-Einstein Bundles on a Compact Kähler Manifold

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In this note we shall give the construction of the moduli space $\mathcal{M}_{\mathrm{HE}}$ of holomorphic irreducible Hermite-Einstein vector bundles on a compact manifold X. This space is introduced as a finite dimensional real analytic subspace of the $\boldsymbol{R}$-Banach analytic manifold of isomorphism classes of irreducible $U(r)$-connections on a hermite vector bundle $E \to X$. The map, which assigns the corresponding semi-connection to a Hermite-Einstein connection descends to a real analytic injective local isomorphism to the complex analytic (not necessarily Hausdorff) moduli space of simple holomorphic vector bundles on X. In particular $\mathcal{M}_{\mathrm{HE}}$ is a Hausdorff, complex space.

A construction of the regular part of $\mathcal{M}_{\mathrm{HE}}$, including differential geometric investigations, was achieved by N. Koiso [5]. Independently M. Lübke and C. Okonek worked on this subject. Our method is a direct generalization of Ito [4].

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Holomorphic vector bundles, whose $\boldsymbol{C}$-endomorphisms consist just of homotheties (constant multiples of the identity), are called *simple*. An immediate consequence of the principles of deformation theory is:

Theorem 1. *Let X be a compact complex manifold, then the set $\mathcal{M}_s$ of isomorphism classes of simple holomorphic vector bundles carries the natural structure of a (not necessarily Hausdorff) complex space.*

The proof follows from a general argument of [3], [6]: If S is a complex space, $s_0 \in S$ a point and $V \to X \times S$ a family of simple holomorphic vector bundles, then all automorphisms of $V_{s_0} = V | X \times \{s_0\}$ can be extended to isomorphisms of the families over a neighborhood of s_0; a fact, which implies the existence of universal deformations; and if $V \to X \times S$ and $W \to X \times R$ are universal families, such that V_{s_0}, $s_0 \in S$ and W_{r_0}, $r_0 \in R$ are isomorphic, then there exists a uniquely determined isomorphism of neighborhoods of r_0 and s_0 resp., which can be lifted to the families of bundles.

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The union of all such S with the above identification clearly has the desired property.

Let $E \to X$ be a fixed differentiable Hermitian vector bundle. It gives rise to a principal G-bundle $P \to X$ with $G = U(r)$. Let $\mathrm{ad}(P) := P \times_G \mathfrak{g}$, with respect to the adjoint representation of G in its Lie algebra $\mathfrak{g}$. If η is a real (or complex) vector bundle on X then $\Omega^p(\eta)$ and $\Omega^{p,q}(\eta)$ resp. shall denote the space of C-sections of $\bigwedge^p T_M \otimes \eta$ and $\bigwedge^{p,q} T_M^C \otimes \eta$ resp. Connections of the G-principal bundle P are certain $\mathfrak{g}$-valued 1-forms on P. These form an affine space $\{A_0 + \alpha;\ \alpha \in \Omega^1(\mathrm{ad}(P))\}$; the curvature $F(A)$ is an element of $\Omega^2(\mathrm{ad}(P))$. The complexification $P^C \to X$ of P is a $\mathrm{GL}(r, C)$-principal bundle, and $\mathrm{ad}(P^C) = P^C \times_{\mathrm{GL}(r)} \mathrm{End}(r, C)$. Alternatively, a connection on E can be described by its covariant derivative $D : \Omega^0(E) \to \Omega^1(E)$. If $D_1 : \Omega^0(E) \to \Omega^1(E)$ is the natural extension of D, the $F(D) = D_1 \circ D \in \Omega^2(\mathrm{End}(E))$. Covariant derivatives $D'' : \Omega^0(E) \to \Omega^{0,1}(E)$ describe so-called *semi-connections*. The equation $0 = D'' \circ D'' \in \Omega^{0,2}(\mathrm{End}(E))$ is an *integrability-condition*, which implies the existence of a unique complex analytic structure on E such that D'' becomes the $\bar\partial$-operator. Integrable semi-connections arise in a natural way as $(0,1)$-parts of $U(r)$-connections, whose curvature $F(D)$ is of type $(1,1)$. The notion of simplicity generalizes to the *irreducibility* of $U(r)$-connections. Given a $U(r)$-connection A one has the following sequence

$$0 \longrightarrow \Omega^0(\mathrm{ad}(P)) \xrightarrow{d_A} \Omega^1(\mathrm{ad}(P)) \xrightarrow{d_A} \Omega^2(\mathrm{ad}(P)) \longrightarrow \cdots$$

where $d_A(\psi) = d\psi + (\psi, A)$. Now A is called *irreducible*, if $d_A : \Omega^0(\mathrm{ad}\,P) \to \Omega^1(\mathrm{ad}\,P)$ is injective.

We equip X with a fixed Kähler form ω, and the inner product on $\Omega^p(\mathrm{ad}\,P)$ given by $\langle \phi, \psi \rangle = -\int \mathrm{trace}(\phi \wedge * \psi)$ gives rise to an H_k^2-Sobolev-structure on $\Omega^p(\mathrm{ad}\,P)$. We choose k sufficiently large and consider connections of class H_k^2. Then $\mathcal{A} := \{A;\ A$ irreducible connection of class $H_k^2\}$ becomes a Banach manifold. This space is acted on differentiably by the gauge group $\mathcal{G} = \mathrm{Aut}(P/X)$ of G-equivariant automorphisms of P over X of class H_{k+1}^2, which is a Banach Lie-group. By methods of Atiyah, Hitchin and Singer [1] we show that the quotient $\mathcal{A}/\mathcal{G}$ is a Hausdorff topological space. Denote by Λ the adjoint of the exterior product by the Kähler form ω, extended to $\mathrm{ad}(P^C)$-valued differential forms. Then a $U(r)$-connection A on P is called *Hermite-Einstein*, if

(i) $F(A)$ is of type $(1,1)$

(ii) $\Lambda F(A) = \lambda \cdot id$ for a $\lambda \in \boldsymbol{R}$.

The holomorphic structure on E induced by the $(0,1)$-part A'' of A is called *Hermite-Einstein-bundle*, and isomorphisms of such holomorphic bundles come from an action of the complexified gauge group $\mathcal{G}^C = \mathrm{Aut}(P^C/X)$. The subspace $\mathcal{A}_{\mathrm{HE}}$ of $\mathcal{A}$ consisting of irreducible Hermite-Einstein connections is a real Banach-analytic subspace (given by a quadratic equation), and is fixed under the gauge group $\mathcal{G}$. As any curvature $F(A)$ can be decomposed into the sum of a trace-free part F^0 and $(1/r)\,\mathrm{Tr}(F)$, where the latter term represents the first Chern-class of E, the connection A is Hermite-Einstein, if F^0 is primitive (i.e. $\Lambda F^0=0$) of type $(1,1)$, and $(1/r)\operatorname{Tr}(F)$ is harmonic. In particular $(1/r)\operatorname{Tr}(F)$ remains unchanged under a continuous variation of A in $\mathcal{A}_{\mathrm{HE}}$. So we have to consider the trace-free part F^0, and the center of $\mathcal{G}$ acts trivially on $\mathcal{A}_{\mathrm{HE}}$ (and $\mathcal{A}$), which implies a reduction to the case $G=SU(r)$, $\mathfrak{g}=\mathfrak{su}(r)$. (One can also achieve this by tensorizing E with a suitable fixed line bundle.) As isomorphism classes of irreducible (Hermite-Einstein) connections are the elements of $\mathcal{M}_{\mathrm{irr}}:=\mathcal{A}/\mathcal{G}$ and $\mathcal{M}_{\mathrm{HE}}:=\mathcal{A}_{\mathrm{HE}}/\mathcal{G}$ resp. we can state:

Theorem 2.

(1) *The moduli space $\mathcal{M}$ of irreducible connections of class H_k^2 is a Hausdorff, real Banach-analytic manifold.*

(2) *The moduli space $\mathcal{M}_{\mathrm{HE}}$ of irreducible Hermite-Einstein connections is a finite dimensional real analytic subspace of $\mathcal{M}$.*

(3) *The assignment $A\to A''$, $A\in\mathcal{A}_{\mathrm{HE}}$ induces an injection $\mathcal{M}_{\mathrm{HE}}\to\mathcal{M}_s$, which is a local isomorphism of real analytic spaces. In particular, the moduli space of irreducible Hermite-Einstein bundles is a Hausdorff complex space.*

We give a sketch of the proof. In all cases, we show a slice-theorem. In (2) we will use an idea of Itoh [4], and in the third part we construct isomorphisms between slices in $\mathcal{A}_{\mathrm{HE}}$ and the Kuranishi slices of holomorphic vector bundles. The injectivity follows by a method of Donaldson [2]: Given A, $A_1\in\mathcal{A}_{\mathrm{HE}}$ such that the induced semi-connections A'', A_1'' differ by an element of $\mathcal{G}^C$, in order to show that A and A_1 are equivalent under the action of the real gauge group, one first performs a reduction from the group $SU(r)$ to the group $H_0^+(r)$ of positive definite, Hermite symmetric matrices of determinant one, and considers the functional m as in [2] (cf. also [4]).

As for the construction of slices for the action of $\mathcal{G}$ and $\mathcal{G}^C$ on $\mathcal{A}$, $\mathcal{A}_{\mathrm{HE}}$ and the space of simple, integrable semi-connections resp., we consider an elliptic complex. Set $\eta=\mathrm{ad}(P)$ and $\Omega_+^2(\eta)=(\Omega^{2,0}(\eta)\oplus\Omega^{0,2}(\eta))_{\boldsymbol{R}}\oplus \boldsymbol{C}\cdot\omega\subset\Omega^2(\eta)$. The map $p^+:\Omega^2(\eta)\to\Omega_+^2(\eta)$ is given by $\gamma\to\langle\gamma,\omega\rangle\cdot\omega$ and on $\Omega^{2,0}(\eta)$, $\Omega^{0,2}(\eta)$ p^+ is the identity. Set $d_A^+=p^+\circ d_A$, then for primitive connections A the following complex is elliptic:

$$0\longrightarrow\Omega^0(\eta)\xrightarrow{d_A}\Omega^1(\eta)\xrightarrow{d_A^+}\Omega_+^2(\eta)\xrightarrow{d_A}(\Omega^{3,0}(\eta)\oplus\Omega^{0,3}(\eta))_{\boldsymbol{R}}\xrightarrow{d_A}\cdots.$$

Denote by d_A^* the formal adjoint of d_A. If $A\in\mathcal{A}$, and ε is a sufficiently small, positive number, a slice for the action of $\mathcal{G}$ near A is

$$U_{A,\varepsilon}=\{A+\alpha\,;\ \alpha\in\Omega^1(\mathrm{ad}\,P),\ d_A^*(\alpha)=0,\ \|\alpha\|<\varepsilon\}.$$

The action $\mathcal{G}\times U_{A,\varepsilon}\to\mathcal{A}$ is a local diffeomorphism, since $d_A:\Omega^0(\eta)\to\Omega^1(\eta)$ is injective. If furthermore A is an irreducible Hermite-Einstein connection, then we consider

$$\mathcal{U}_{A,\varepsilon}:=\mathcal{A}_{\mathrm{HE}}\cap U_{A,\varepsilon}=\{A+\alpha\in U_{A,\varepsilon}\,;\ d_A^+\alpha=\alpha\sharp\alpha\},\qquad \alpha\sharp\alpha:=p^+(\alpha\wedge\alpha).$$

We use Kuranishi's method to show that $\mathcal{U}_{A,\varepsilon}$ is finite dimensional and analytic. The standard slice for the $\mathcal{G}^C$-action on the space of all integrable, simple semi-connections (which induces the universal deformation) consists of

$$V_{A'',\varepsilon}=\{A''+\beta\in\Omega^{0,1}(\mathrm{ad}(P^C))\,;\ \bar\partial_A\beta=\beta\wedge\beta,\ \bar\partial_A^*\beta=0\}.$$

However, the canonical map $\Omega^1(\mathrm{ad}(P))\to\Omega^{0,1}(\mathrm{ad}(P^C))$ does not map $\mathcal{U}_{A,\varepsilon}$ to $V_{A'',\varepsilon}$. We construct an analytic isomorphism from $V_{A''}$, to

$$\tilde V_{A'',\varepsilon}:=\Big\{A''+\alpha''\,;\ \bar\partial_{A''}\alpha''-\alpha''\wedge\alpha''=0,\quad \bar\partial_{A''}^*\alpha''+\frac{i}{2}\Lambda(\alpha'\wedge\alpha''+\alpha''\wedge\alpha')=0,\ \alpha'=-{}^t\alpha''\Big\}:$$

the latter slice now is canonically isomorphic to the (in general singular) slice $\mathcal{U}_{A,\varepsilon}$.

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