13. On the Relative CR Structure

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Let $\overline{M}$ be an m-dimensional compact complex manifold with smooth boundary M_0, and let $\overline{N}$ be an n-dimensional closed submanifold of $\overline{M}$ with boundary N_0. The purpose of this note is to report the possibility of extending deformations of the relative CR structure on the pair (M_0, N_0) to deformations of the relative complex structure on the pair $(\overline{M}, \overline{N})$ in the sense of [3]. The details will appear elsewhere.

§1. **Preliminaries.**

(1.1) Suppose that $\overline{M}$ is the closure of a relatively compact open subset M of a C^∞ manifold M' in which M has a smooth boundary M_0. Let N' be a C^∞ submanifold of M' such that $\overline{N}=\overline{M}\cap N'$ and $N_0=M_0\cap N'$. Let $\Xi_{\overline{N}/\overline{M}}$ be the sheaf of germs of holomorphic vector fields on $\overline{M}$ which are tangential to $\overline{N}$ at each point of $\overline{N}$. Let $T'\overline{M}$ (resp. $T''\overline{M}$) be the holomorphic (resp. the antiholomorphic) tangent bundle of $\overline{M}$. Denote the complexification of the tangent bundle TM_0 of M_0 by CTM_0. Set ${}^oT'=(T'\overline{M}|_{M_0})\cap CTM_0$ and ${}^oT''=\overline{{}^oT'}$. Then the complex fiber dimension of ${}^oT'$ is $m-1$, and we have the direct sum decomposition $CTM_0={}^oT'\oplus{}^oT''\oplus F$, where F is the complexification of a real one-dimensional subbundle of CTM_0.

(1.2) Let $\overline{M}$ (resp. $\overline{N}$) be the underlying C^∞ manifold of $\overline{M}$ (resp. $\overline{N}$). Because of the technical reason, we fix a real analytic totally geodesic Riemannian metric on $\overline{M}$ such that $\overline{N}$ is a totally geodesic submanifold of $\overline{M}$ [3]. Let U be a coordinate neighborhood in $\overline{M}$ with coordinates $z=(z^1,\cdots,z^m)$. If $U\cap\overline{N}\neq\phi$, then $(z^{(n)},0):=(z^1,\cdots,z^n,\overbrace{0,\cdots,0}^{m-n})$ gives local coordinates on $\overline{N}$. By using the above metric, we denote a C^∞ map $h:M'\to\boldsymbol{R}$ such that $|h(z)|$ is geodesic distance from z to M_0. Then $M=\{z\in M'\mid h(z)<0\}$, $M_0=\{z\in M'\mid h(z)=0\}$ and $dh\neq0$ on M_0. As a purely imaginary generator of F, we may choose an element $P=P'-P''$ such that $P'=\sum_{k=1}^m p^k\partial/\partial z^k\in\Gamma(M_0,\Xi_{\overline{N}/\overline{M}}|_{M_0})$, $P''=\overline{P'}$ and $dh(P')=dh(P'')=1$. Note that $(P'-P'')|_{N_0}\in\Gamma(N_0,(F|_{N_0})\cap CTN_0)$.

(1.3) We set $h_j=\partial h/\partial z^j$ and $h_{\bar{j}}=\partial h/\partial\bar{z}^j$. On $U_0=U\cap M_0$ we denote $Z_j=\partial/\partial z^j-h_jP'$, $Z_{\bar{j}}=\overline{Z}_j$ for $1\leq j\leq m$. Then $Z_1,\cdots,Z_m$ (resp. $Z_{\bar{1}},\cdots,Z_{\bar{m}}$) generate ${}^oT'$ (resp. ${}^oT''$) over U_0. Let $i:M_0\hookrightarrow M'$ be the injection. Put $\overline{Z}^k=i^*d\bar{z}^k-\bar{p}^k i^*\bar{\partial}h$ for $1\leq k\leq m$. Then $\overline{Z}^1,\cdots,\overline{Z}^m$ generate ${}^oT''^*$ over U_0. Let $\phi\in\Gamma(M_0,\wedge^q\,{}^oT''^*\otimes(T'\overline{M}|_{M_0}))$. Then ϕ is written in the form

$$\phi=\sum_{\alpha_j,\beta=1}^m\phi^\beta_{\alpha_1\cdots\alpha_q}\overline{Z}^{\alpha_1}\wedge\cdots\wedge\overline{Z}^{\alpha_q}(\partial/\partial z^\beta).$$

We call it $T'\overline{M}|_{M_0}$-valued differential form of type $(0,q)_b$. We also define the tangential Cauchy-Riemann operator $\bar{\partial}_b$ by $\bar{\partial}_b=\sum_{k=1}^m(\partial/\partial\bar{z}^k)\overline{Z}^k$.

§2. A fine resolution of $\Xi_{N/M}|_{M_0}$.

(2.1) Let $(\mathcal{B}^{0,q}, \bar{\partial})_{q=0}^{m}$ be a fine resolution of $\Xi_{N/M}$ over $\bar{M}$ [3]. We consider a fine resolution of $\Xi_{N/M}|_{M_0}$ over M_0 in below. Suppose that $\Theta_{\bar{M}}$ is the sheaf of germs of holomorphic vector fields on $\bar{M}$, and that Ψ is the sheaf over $\bar{N}$ of germs of holomorphic sections of the normal bundle of $\bar{N}$ in $\bar{M}$. Then we have the following exact sequence over $M_0: 0 \to \Xi_{N/M}|_{M_0} \to \Theta_{\bar{M}}|_{M_0} \to \Psi|_{M_0} \to 0$. Let $\mathcal{A}^{0,q}(T'\bar{M}|_{M_0})$ (resp. $\mathcal{A}^{0,q}(\Psi|_{M_0})$) be the sheaf of germs of $T'\bar{M}|_{M_0}$-valued (resp. $\Psi|_{M_0}$-valued) differential form of type $(0,q)_b$ on M_0 (resp. N_0). Then $(\mathcal{A}^{0,q}(T'\bar{M}|_{M_0}), \bar{\partial}_b)_{q=0}^{m}$ (resp. $(\mathcal{A}^{0,q}(\Psi|_{M_0}), \bar{\partial}_b)_{q=0}^{m}$) is a fine resolution of $\Theta_{\bar{M}}|_{M_0}$ (resp. $\Psi|_{M_0}$). Put $\mathcal{B}_o^{0,q} = \mathrm{Ker}\{\mathcal{A}^{0,q}(T'\bar{M}|_{M_0}) \to \mathcal{A}^{0,q}(\Psi|_{M_0})\}$. Then it is easily verified that $(\mathcal{B}_o^{0,q}, \bar{\partial}_b)_{q=0}^{m}$ becomes a desired *fine resolution of* $\Xi_{N/M}|_{M_0}$ *over* M_0.

(2.2) We give the following definitions of subsheaves of $\mathcal{B}^{0,q}$ and $\mathcal{B}_o^{0,q}$ used in next section.

Definition 2.1. Let $C_{\bar{M}}^{\infty}$ be the sheaf of germs of C^{∞} functions on $\bar{M}$. Denote $(\mathcal{J}_{\bar{N}})$ by an ideal generated by the ideal sheaf $\mathcal{J}_{\bar{N}}$ of $\bar{N}$ in $C_{\bar{M}}^{\infty}$. Then $\widetilde{\mathcal{B}^{0,q}}$ is defined as follows:

$$\widetilde{\mathcal{B}^{0,q}} = \{\phi = \textstyle\sum_{\alpha_j,\beta=1}^{m} \phi_{\alpha_1\cdots\alpha_q}^{\beta}(z) d\bar{z}^{\alpha_1} \wedge \cdots \wedge d\bar{z}^{\alpha_q}(\partial/\partial z^{\beta}) \in \mathcal{B}^{0,q} \mid \phi_{\alpha_1\cdots\alpha_q}^{\beta}(z) \in (\mathcal{J}_{\bar{N}}) \text{ for } 1 \leqq {}^{\forall}\alpha_j \leqq m \text{ and } n+1 \leqq \beta \leqq m\}.$$

Definition 2.2. Let $C_{M_0}^{\infty}$ be the sheaf of germs of C^{∞} functions on M_0. Set $\mathcal{O}_{M_0} = \{f \in C_{M_0}^{\infty} \mid \bar{\partial}_b f = 0\}$ and $\mathcal{J}_{N_0} = \{f \in \mathcal{O}_{M_0} \mid f|_{N_0} = 0\}$. Denote $(\mathcal{J}_{N_0})$ by an ideal generated by the sheaf $\mathcal{J}_{N_0}$ in $C_{M_0}^{\infty}$. Then $\widetilde{\mathcal{B}_o^{0,q}}$ is defined as follows:

$$\widetilde{\mathcal{B}_o^{0,q}} = \{\phi = \textstyle\sum_{\alpha_j,\beta=1}^{m} \phi_{\alpha_1\cdots\alpha_q}^{\beta} \bar{Z}^{\alpha_1} \wedge \cdots \wedge \bar{Z}^{\alpha_q}(\partial/\partial z^{\beta}) \in \mathcal{B}_o^{0,q} \mid \phi_{\alpha_1\cdots\alpha_q}^{\beta} \in (\mathcal{J}_{N_0}) \text{ for } 1 \leqq {}^{\forall}\alpha_j \leqq m \text{ and } n+1 \leqq \beta \leqq m\}.$$

§3. A relative *CR* structure.

(3.1) Let us recall the definition of a relative complex structure on $(\bar{M}, \bar{N})$. First a *relative almost complex structure* is defined by the pair $(T'', \mathcal{T}'')$, where T'' is a subbundle of $CT\bar{M}$ of fiber dimension m and $\mathcal{T}''$ is a subsheaf of the sheaf over $\bar{M}$ of germs of C^{∞} sections of T''. Further the above pair can be parametrized by $\omega \in \Gamma(\bar{M}, \widetilde{\mathcal{B}^{0,1}})$. We denote this by $(T''_{\omega}, \mathcal{T}''_{\omega})$. Suppose that $\omega \in \Gamma(\bar{M}, \widetilde{\mathcal{B}^{0,1}})$ is sufficiently near to zero in C^0-topology. Then $(T''_{\omega}, \mathcal{T}''_{\omega})$ is a *relative complex structure* if and only if $\Omega(\omega) \equiv \bar{\partial}\omega - (1/2)[\omega, \omega] = 0$ (for more details, we refer to [3]).

(3.2) We now give the following

Definition 3.1. By a *relative almost CR structure* on (M_0, N_0) we mean the triple $({}^oE'', G, {}^o\mathcal{T}'')$ which satisfies the following conditions:

(1) ${}^oE''$ (resp. G) is a subbundle of CTM_0 of fiber dimension $m-1$ (resp. 1),

(2) ${}^o\mathcal{T}''$ is a subsheaf of the sheaf over M_0 of germs of C^{∞} sections of ${}^oE''$,

(3) $CTM_0 = {}^oE'' \oplus {}^oE' \oplus G$, ${}^oE' = \overline{{}^oE''}$ and $\mathcal{T}(M_0, N_0) = {}^o\mathcal{T}'' \oplus {}^o\mathcal{T}'$, ${}^o\mathcal{T}' = \overline{{}^o\mathcal{T}''}$ where $\mathcal{T}(M_0, N_0) = \widetilde{\overline{\mathcal{B}_o^{0,0}}} \oplus \widetilde{\mathcal{B}_o^{0,0}}$.

Further if the Lie bracket $[L, L']$ of any two sections L, L' of ${}^oE''$ (resp.

${}^{o}\mathcal{T}''$) over an open set of M_0 is also a section of ${}^{o}E''$ (resp. ${}^{o}\mathcal{T}''$), we say (${}^{o}E'', G, {}^{o}\mathcal{T}''$) a *relative CR structure.*

Let ρ be the canonical isomorphism from $T'\overline{M}|_{M_0}$ to ${}^{o}T'\oplus F$ defined by $\rho(\partial/\partial z^j)\equiv\partial^\rho/\partial z^j=Z_j+h_jP$ for $1\leq j\leq m$. Now we suppose that $\varphi: {}^{o}T''\to T'\overline{M}|_{M_0}$ is a bundle map such that $\rho\circ\varphi(\widetilde{\overline{\mathcal{B}_o^{0,0}}})\subseteq\widetilde{\mathcal{B}_o^{0,0}}$. Then any relative almost CR structure sufficiently close to (${}^{o}T'', F, \widetilde{\overline{\mathcal{B}_o^{0,0}}}$) is the graph of maps $\rho\circ\varphi: {}^{o}T''\to {}^{o}T'\oplus F$ and $\rho\circ\varphi|_{\widetilde{\overline{\mathcal{B}_o^{0,0}}}}: \widetilde{\overline{\mathcal{B}_o^{0,0}}}\to\widetilde{\mathcal{B}_o^{0,0}}$. Denote this relative almost CR structure on (M_0, N_0) by (${}^{o}T''_\varphi, F_\varphi, \widetilde{\overline{\mathcal{B}_{o\varphi}^{0,0}}}$). Thus relative almost CR structures on (M_0, N_0) sufficiently close to (${}^{o}T'', F, \widetilde{\overline{\mathcal{B}_o^{0,0}}}$) are parametrized by elements of $\Gamma(M_0, \widetilde{\mathcal{B}_o^{0,1}})$. For a sufficiently small φ in $\Gamma(M_0, \widetilde{\mathcal{B}_o^{0,1}})$, (${}^{o}T''_\varphi, F_\varphi, \widetilde{\overline{\mathcal{B}_{o\varphi}^{0,0}}}$) is a relative CR structure if and only if $\Phi(\varphi)=\sum_{j=1}^m \Phi^j(\varphi)\partial/\partial z^j$ vanishes identically. Here $\Phi^j(\varphi)=\bar\partial_b\varphi^j-\sum_{\alpha,\beta=1}^m(\partial^\rho\varphi_\alpha^j/\partial z^\beta)\varphi^\beta\wedge\bar Z^\alpha+\varphi(h)\wedge\sum_{\alpha=1}^m\varphi_\alpha^j\{\bar\partial_b\bar p^\alpha-\sum_{\beta=1}^m(\partial^\rho\bar p^\alpha/\partial z^\beta)\varphi^\beta\}$, where $\varphi^\beta=\sum_{\alpha=1}^m\varphi_\alpha^\beta\bar Z^\alpha$.

(3.3) Now we show the important relation between a relative complex structure and a relative CR structure.

Key lemma. *Let* $\omega\in\Gamma(\overline{M}, \widetilde{\mathcal{B}^{0,1}})$ *and* $\varphi\in\Gamma(M_0, \widetilde{\mathcal{B}_o^{0,1}})$. *Then* $(T''_\omega|_{M_0})\cap CTM_0={}^{o}T''_\varphi$ *and* $(\mathcal{T}''_\omega|_{M_0})\cap C^\infty({}^{o}T''_\varphi)={}^{o}\mathcal{T}''_\varphi$ *are satisfied if and only if* $\tau_\varphi(\omega)\equiv\tau(\omega)+\varphi(h)\nu(\omega)=\varphi$, *where* $i^*\omega=\tau(\omega)+\nu(\omega)\wedge i^*\bar\partial h$ *and* $\mathcal{T}''_\varphi=\widetilde{\overline{\mathcal{B}_{o\varphi}^{0,0}}}$.

§ 4. Extending problem of relative CR structures.

(4.1) Suppose that $(T''_\omega, \mathcal{T}''_\omega)$ is a relative complex structure. We set ${}^{o}E''=(T''_\omega|_{M_0})\cap CTM_0$ and ${}^{o}\mathcal{T}''=(\mathcal{T}''_\omega|_{M_0})\cap C^\infty({}^{o}E'')$, where $C^\infty({}^{o}E'')$ is the sheaf of germs of C^∞ sections of ${}^{o}E''$ over M_0. Then (${}^{o}E'', G, {}^{o}\mathcal{T}''$) is a relative CR structure, where $CTM_0={}^{o}E''\oplus{}^{o}E'\oplus G$, ${}^{o}E'={}^{o}\overline{E}''$. The converse of this statement for deformations of relative CR structures gives an extension problem mentioned in the preface. Let φ be in a neighborhood of zero in $\Gamma(M_0, \widetilde{\mathcal{B}_o^{0,1}})$ in some Sobolev norm topology and (${}^{o}T''_\varphi, F_\varphi, {}^{o}\mathcal{T}''_\varphi$) is a relative CR structure. Then the question is to find $\omega\in\Gamma(\overline{M}, \widetilde{\mathcal{B}^{0,1}})$ such that $(T''_\omega|_{M_0})\cap CTM_0={}^{o}T''_\varphi$, $(\mathcal{T}''_\omega|_{M_0})\cap C^\infty({}^{o}T''_\varphi)={}^{o}\mathcal{T}''_\varphi$ and $(T''_\omega, \mathcal{T}''_\omega)$ is a relative complex structure.

In view of Key lemma this problem reduces to the non-linear boundary value problem $\Omega(\omega)=0$, $\tau_\varphi(\omega)=\varphi$ with the necessary condition $\Phi(\varphi)=0$.

(4.2) We now state here our problems.

Problem I. Can any deformations of the relative CR structure on (M_0, N_0) be directly extended to deformations of $(\overline{M}, \overline{N})$?

As a special case of this problem, we ask

Problem I*. Does there exist the versal family (in the sense of [3]) of deformations of $(\overline{M}, \overline{N})$ which leaves the relative CR structure on (M_0, N_0) fixed?

§ 5. Main theorem.

In this section, we shall give an answer to Problems I and I* discussed in § 4. For this purpose, we make the following

Definition 5.1. The pair (M_0, N_0) of the boundaries satisfies *condition* $Y(q)$ if at every point of M_0 (resp. N_0) the Levi form of h (resp. $h|_{\bar{N}}$) on $\bar{M}$ (resp. $\bar{N}$) has at least q (resp. q) positive eigenvalues, where h is the C^∞ map as defined in (1.2).

Then we state here our main

Theorem. (I) *Suppose that* $\dim_C \bar{N} \geqq 2$, *the pair* (M_0, N_0) *satisfies condition* $Y(2)$ *and* $H_c^2(M, \Xi_{\bar{N}/\bar{M}})=0$, *where* $H_c^2(M, \Xi_{\bar{N}/\bar{M}})$ *is the second compactly supported cohomology group with coefficients in* $\Xi_{\bar{N}/\bar{M}}$. *If* φ *is a sufficiently small element in* $\Gamma(M_0, \widetilde{\mathscr{B}_0^{0,1}})$ *with* $\Phi(\varphi)=0$, *then one can find* $\omega \in \Gamma(\bar{M}, \widetilde{\mathscr{B}^{0,1}})$ *such that* $\Omega(\omega)=0$ *and* $\tau_\varphi(\omega)=\varphi$.

(I*) *Suppose that* $\dim_C \bar{N} \geqq 1$ *and condition* $Y(1)$ *is satisfied. Then there exists the versal family of deformations of* $(\bar{M}, \bar{N})$ *leaving the relative CR structure on* (M_0, N_0) *fixed.*

First we notice that the proof of Theorem is reduced to the case $\operatorname{codim}_C \bar{N}=1$ by using the monoidal transformation of $\bar{M}$ with the center $\bar{N}$ (cf. [2]). In this case we can show that by applying the method in our previous paper [3], the Kohn-Morrey basic estimate [1, 4] and a Nash-Moser type inverse mapping theorem [5].

References

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