114. *Dual Pairs on Spinors*

Cases of (C_m, C_n) *and* $(C_m^{(1)}, C_n^{(1)})$

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§0. **Introduction.** Weyl's reciprocity theorem says that the symmetric group $\mathfrak{S}_m$ and the general linear group $GL(n, \boldsymbol{C})$ are mutually commutant (i.e. $(\mathfrak{S}_m, GL(n, \boldsymbol{C}))$ forms a *dual pair* [3]) on the tensor space $(\boldsymbol{C}^n)^{\otimes m}$. The purpose of this paper is to give the spinor analogues of this theorem: we claim $(\mathfrak{sp}(2m), \mathfrak{sp}(2n))$ forms a dual pair on the $\mathfrak{o}(4mn)$-module $\wedge(\boldsymbol{C}^{2mn})$, and describe its irreducible decomposition as a $\mathfrak{sp}(2m)\oplus\mathfrak{sp}(2n)$-module (Theorem A). The affine Lie algebra pair $(C_m^{(1)}, C_n^{(1)})$ also forms a dual pair on $\wedge(\hat{W}_{4mn}^{-})$ (Theorem B). As corollaries we deduce new dualities for branching rules. Details appear in our forthcoming paper [2], where we also construct various dual pairs for all classical Lie algebras, and for their affinizations. Our method is similar to that of [3], which deals with dual pairs on the Shale-Weil modules.

§1. **Finite dimensional case.** **1.1.** After [1] we review the spinor representation of the orthogonal Lie algebra $\mathfrak{o}(2l)=\left\{X\in\mathfrak{gl}(2l)\,\middle|\,{}^tX\begin{bmatrix}0 & 1_l\\ 1_l & 0\end{bmatrix}+\begin{bmatrix}0 & 1_l\\ 1_l & 0\end{bmatrix}X=0\right\}$. Let $\mathcal{C}(W_{2l})$ be the Clifford algebra over the vector space $W_{2l}:=V^l\oplus V_l\simeq\boldsymbol{C}^{2l}$, where $V^l:=\oplus_{j=1}^l\boldsymbol{C}\psi^j$ and $V_l:=\oplus_{j=1}^l\boldsymbol{C}\psi_j$, with a symmetric bilinear form $(\ ,\ )$ defined by

$$(\psi^i, \psi_j)=\delta^i{}_j \quad\text{and}\quad (\psi^i, \psi^j)=0=(\psi_i, \psi_j) \quad\text{for } 1\le i,\, j\le l.$$

As a $\boldsymbol{C}$-algebra $\mathcal{C}(W_{2l})\simeq\mathrm{Mat}(2^l, \boldsymbol{C})$. Its irreducible representation is realized on the exterior algebra $\wedge(V^l)$, with defining 1 the vacuum vector and V^l (resp. V_l) the creation (resp. annihilation) operators. Write $[a, b]$ for $ab-ba$, and the spinor representation s is defined by

$$s:\mathfrak{o}(2l)\ni\begin{bmatrix}E^i{}_j & 0\\ 0 & -E^j{}_i\end{bmatrix}\longmapsto\frac{1}{2}[\psi^i, \psi_j]\in\mathcal{C}(W_{2l})\simeq\mathrm{End}\wedge(V^l),$$

$$\begin{bmatrix}0 & E^i{}_j-E^j{}_i\\ 0 & 0\end{bmatrix}\longmapsto\frac{1}{2}[\psi^i, \psi^j],\quad \begin{bmatrix}0 & 0\\ E^i{}_j-E^j{}_i & 0\end{bmatrix}\longmapsto\frac{1}{2}[\psi_i, \psi_j]\qquad(1\le i, j\le l).$$

1.2. Now we deal with the dual pair $(\mathfrak{sp}(2m), \mathfrak{sp}(2n))$. Recall that

$$\mathfrak{sp}(2n):=\left\{X\in\mathfrak{gl}(2n)\,\middle|\,{}^tX\begin{bmatrix} & 1_n\\ -1_n & \end{bmatrix}+\begin{bmatrix} & 1_n\\ -1_n & \end{bmatrix}X=0\right\}$$
$$=\left\{\begin{bmatrix}A & B\\ C & -{}^tA\end{bmatrix}\,\middle|\,\begin{matrix}A, B, C\in\mathfrak{gl}(n)\\ {}^tB=B,\ {}^tC=C\end{matrix}\right\},$$

and let $l=2mn$. Then there exist two Lie algebra monomorphisms $R:\mathfrak{sp}(2n)\to\mathfrak{o}(2l)$ and $L:\mathfrak{sp}(2m)\to\mathfrak{o}(2l)$ so that $R(\mathfrak{sp}(2n))'=L(\mathfrak{sp}(2m))$ and $L(\mathfrak{sp}(2m))'$

$=R(\mathfrak{sp}(2n))$, where $A':=\{x \in \mathfrak{o}(2l) \mid [x, A]=0\}$. Such R (resp. L) arises from the right (resp. left) action of $\mathfrak{sp}(2n)$ (resp. $\mathfrak{sp}(2m)$) on $\mathrm{Mat}(2m\times 2n, \boldsymbol{C})$. Considering $s\circ R$ and $s \circ L$, we get

Proposition 1. *The map* $\mathfrak{sp}(2n)\to C(W_{4mn})\simeq \mathrm{End} \wedge(V^{2mn})$ *defined by*

$$\mathfrak{sp}(2n) \ni \begin{bmatrix} E^i{}_j & 0 \\ 0 & -E^j{}_i \end{bmatrix} \longmapsto \frac{1}{2}\sum_{p=1}^{m} ([\psi^{i,p}, \psi_{j,p}]-[\psi^{-j,p}, \psi_{-i,p}]) \in \mathrm{End} \wedge (V^{2mn})$$

$$\begin{bmatrix} 0 & E^i{}_j+E^j{}_i \\ 0 & 0 \end{bmatrix} \longmapsto \frac{1}{2}\sum_{p=1}^{m} ([\psi^{i,p}, \psi_{-j,p}]+[\psi^{j,p}, \psi_{-i,p}])$$

$$\begin{bmatrix} 0 & 0 \\ E^i{}_j+E^j{}_i & 0 \end{bmatrix} \longmapsto \frac{1}{2}\sum_{p=1}^{m} ([\psi^{-i,p}, \psi_{j,p}]+[\psi^{-j,p}, \psi_{i,p}]) \qquad (1\leq i, j\leq n)$$

is a Lie algebra monomorphim, and so is the map

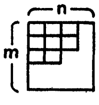
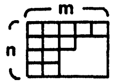
$$\mathfrak{sp}(2m) \ni \begin{bmatrix} E^p{}_q & 0 \\ 0 & -E^q{}_p \end{bmatrix} \longmapsto \frac{1}{2}\sum_{j=1}^{n} ([\psi^{j,p}, \psi_{j,q}]+[\psi^{-j,p}, \psi_{-j,q}]) \in \mathrm{End} \wedge (V^{2mn})$$

$$\begin{bmatrix} 0 & E^p{}_q+E^q{}_p \\ 0 & 0 \end{bmatrix} \longmapsto \frac{1}{2}\sum_{j=1}^{n} ([\psi^{j,p}, \psi^{-j,q}]+[\psi^{j,q}, \psi^{-j,p}])$$

$$\begin{bmatrix} 0 & 0 \\ E^p{}_q+E^q{}_p & 0 \end{bmatrix} \longmapsto \frac{1}{2}\sum_{j=1}^{n} ([\psi_{j,p}, \psi_{-j,q}]+[\psi_{j,q}, \psi_{-j,p}]) \qquad (1\leq p, q\leq m).$$

Here we use symbols $\psi^{\pm k,p}$ (resp. $\psi_{\pm k,p}$) $(1\leq p\leq m,\ 1\leq k\leq n)$ for the basis of V^l (resp. V_l), instead of ψ^j (resp. ψ_j) $(1\leq j\leq l)$ in **1.1.** Note that $\wedge(V^l)\simeq \wedge(V^{2n})^{\otimes m}\simeq \wedge(\boldsymbol{C}^{2n})^{\otimes m}$ as $\mathfrak{sp}(2n)$-modules.

Let $\mathfrak{h}:=\{h \in \mathfrak{sp}(2m) \mid h \text{ is diagonal}\}$, $\mathfrak{n}_+:=\left\{\begin{bmatrix} A & B \\ 0 & -{}^tA\end{bmatrix} \middle| A \text{ is strictly upper triangular, } {}^tB=B\right\}$ and $\mathfrak{n}_-:={}^t\mathfrak{n}_+$, then we fix a triangular decomposition $\mathfrak{sp}(2\mathrm{n})=\mathfrak{n}_+\oplus\mathfrak{h}\oplus\mathfrak{n}_-$. By $L^{\mathfrak{g}}(\lambda)$ we denote the irreducible $\mathfrak{g}$-module with highest weight λ. A sequence $Y=(y_1, \cdots, y_m) \in \boldsymbol{Z}^m$ with $n\geq y_1\geq \cdots \geq y_m\geq 0$ is called a Young diagram contained in the $m\times n$ rectangle, and the set of all such sequences is denoted by R_{mn}. For $Y=(y_i) \in \mathrm{R}_{mn}$ we set $|Y|:=\Sigma_i y_i$, and define $Y^\dagger \in \mathrm{R}_{nm}$ by taking the complement of Y in R_{mn} and transposing it: for example,

$$\mathrm{R}_{mn} \ni Y=\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{bmatrix} 3 \\ 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} =$$  $$\longmapsto Y^\dagger=\begin{pmatrix} y^\dagger_1 \\ \vdots \\ y^\dagger_n \end{pmatrix} :=$$  $$= \begin{bmatrix} 5 \\ 3 \\ 2 \\ 2 \end{bmatrix} \in \mathrm{R}_{nm}$$

$(m=5,\ n=4)$.

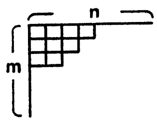
Put $Y\left(\begin{bmatrix} E^j{}_j & \\ & -E^j{}_j\end{bmatrix}\right):=y_j$ for $Y=(y_1, \cdots, y_m) \in \mathrm{R}_{mn}$, and we identify a diagram with a dominant integral weight of $\mathfrak{sp}(2m)$. Then our main result is

Theorem A. *As a* $\mathfrak{sp}(2m)\oplus\mathfrak{sp}(2n)$*-module,*

$$\wedge(V^{2mn})\simeq \oplus_{Y \in \mathrm{R}_{mn}} L^{\mathfrak{sp}(2m)}(Y)\otimes L^{\mathfrak{sp}(2n)}(Y^\dagger).$$

The highest weight vector of the Y-component with respect to $\mathfrak{sp}(2m)\oplus\mathfrak{sp}(2n)$ *is* $\wedge_{\tilde{y}^p_j=1}\psi^{p,j}\cdot 1$, *where* $Y \in \mathrm{R}_{mn}$ *is identified with* $(\tilde{y}^p_j)$ *as follows:*

$\mathrm{R}_{mn} \ni Y$

= 

$$\longleftrightarrow \begin{array}{c} \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ m \end{array} \begin{array}{c} 1\cdots n;\ -n\cdots -1 \\ \left[\begin{array}{ll} 1\cdots 1 & 11110\cdot \\ & 1110\cdot\cdot \\ \cdots & 110\cdots \\ & 0\ \cdots\cdot \\ 1\cdots 1 & \cdots\cdot 0 \end{array}\right] \end{array} = (\tilde{y}^p_j)_{\substack{p=1;\cdots;m \\ j=\pm 1;\cdots;\pm n}} \in \mathrm{Mat}(m\times 2n).$$

All irreducible highest weight module of $\mathfrak{sp}(2n)$ *appears as the irreducible component of* $\wedge(V^{2mn})$, *when m varies over* $\boldsymbol{Z}_{>0}$.

§ 2. **Affine version.** **2.1.** Let $\hat{\mathfrak{g}} := \mathfrak{g}\otimes \boldsymbol{C}[t, t^{-1}]\oplus \boldsymbol{C}c^{\hat{\mathfrak{g}}}$ be the non-twisted affine Lie algebra associated to a simple Lie algebra $\mathfrak{g}$ (see [5]). We review Frenkel's spinor representation of $\mathfrak{o}\hat{}(2l)$ [1]. Let $\mathcal{C}(\hat{W}_{2l})$ be the Clifford algebra over the $\boldsymbol{C}$-vector space $\hat{W}_{2l} := W_{2l}\otimes \boldsymbol{C}[t, t^{-1}]$ with a symmetric form $(\psi(\mu), \psi'(\mu')) := \delta_{\mu+\mu',0}(\psi, \psi')$, where $\psi(\mu) := \psi\otimes t^{\mu}$. Put $\hat{W}^+_{2l} := (W_{2l}\otimes t\boldsymbol{C}[t])\oplus (V_l\otimes 1)$ and $\hat{W}^-_{2l} := (W_{2l}\otimes t^{-1}\boldsymbol{C}[t^{-1}])\oplus(V^l\otimes 1)$, then $\wedge(\hat{W}^-_{2l})$ becomes an irreducible $\mathcal{C}(\hat{W}_{2l})$-module by defining 1 the vacuum, $\hat{W}^-_{2l}$ (resp. $\hat{W}^+_{2l}$) the creation (resp. annihilation) operators. Define

$$:a(\mu)b(\nu): \ := a(\mu)b(\nu) - (a(\mu), b(\nu))\varepsilon, \text{ where } \varepsilon := \begin{cases} 1 & \text{if } \mu > 0 > \nu, \\ 1/2 & \text{if } \mu = 0 = \nu, \\ 0 & \text{otherwise} \end{cases}$$

and the spinor representation $\hat{s}$ is defined by

$$\hat{s} : \hat{\mathfrak{o}}(2l) \ni c^{\mathfrak{o}\hat{}(2l)} \longmapsto id \in \mathrm{End} \wedge(\hat{W}^-_{2l})$$
$$X(k) := X\otimes t^k \longmapsto \Sigma_{\mu\in\boldsymbol{Z}} :a(\mu)b(k-\mu):,$$

for $X\in\mathfrak{o}(2l)$ that satisfies $s(X) = (1/2)[a, b]$ $(a, b\in W_{2l})$ and $k\in\boldsymbol{Z}$.

2.2. We proceed to the dual pair $(\mathrm{C}^{(1)}_m, \mathrm{C}^{(1)}_n)$. Again taking $l=2mn$ we get a Lie algebra monomorphism $\hat{L}$ (resp. $\hat{R}$) by defining

$$\hat{L} : \mathfrak{sp}\hat{}(2m) \ni A(k) \longmapsto L(A)(k) \in \mathfrak{o}\hat{}(2l) \quad \left(\begin{array}{l} \text{resp. } \hat{R} : \mathfrak{sp}\hat{}(2n) \ni A(k) \\ \qquad\qquad \longmapsto R(A)(k) \in \mathfrak{o}\hat{}(2l) \\ c^{\mathfrak{sp}\hat{}(2n)} \longmapsto m\cdot c^{\mathfrak{o}\hat{}(2l)} \end{array}\right)$$
$$c^{\mathfrak{sp}\hat{}(2m)} \longmapsto n\cdot c^{\mathfrak{o}\hat{}(2l)}$$

Proposition 2. *The map* $\hat{s}\circ\hat{L} : \mathfrak{sp}\hat{}(2m) \to \mathrm{End}\wedge(\hat{W}^-_{4mn})$ (*resp.* $\hat{s}\circ\hat{R} : \mathfrak{sp}\hat{}(2n) \to \mathrm{End}\wedge(\hat{W}^-_{4mn})$) *is a level n* (*resp. m*) *integrable representation of* $\mathfrak{sp}\hat{}(2m)$ (*resp.* $\mathfrak{sp}\hat{}(2n)$), *and* $[\hat{s}\circ\hat{L}(\mathfrak{sp}\hat{}(2m)), \hat{s}\circ\hat{R}(\mathfrak{sp}\hat{}(2n))] = 0$.

Theorem B. *As a* $\mathfrak{sp}\hat{}(2m)\oplus\mathfrak{sp}\hat{}(2n)$*-module,*

$$\wedge(\hat{W}^-_{4mn}) \simeq \oplus_{Y\in\mathrm{R}_{mn}} L^{\mathfrak{sp}\hat{}(2m)}(Y, n)\otimes L^{\mathfrak{sp}\hat{}(2n)}(Y^{\dagger}, m).$$

The highest weight vector of the Y-component is $\wedge_{\tilde{y}^p_j=1}\psi^{p,j}(0)\cdot 1$, *where* $(\tilde{y}^p_j)$ *is as in Theorem A. All level m irreducible integrable highest weight module of* $\mathrm{C}^{(1)}_n$ *appears as the irreducible component of* $\wedge(\hat{W}^-_{4mn})$.

Here we write $L^{\hat{\mathfrak{g}}}(Y, n)$ for $L^{\hat{\mathfrak{g}}}(\lambda)$, when the highest weight $\lambda\in(\mathfrak{h}\otimes 1 \oplus\boldsymbol{C}c^{\hat{\mathfrak{g}}})^*$ satisfies $\lambda(c^{\hat{\mathfrak{g}}}) = n$ and $\lambda|_{\mathfrak{h}\otimes 1} = Y\in\mathrm{R}_{mn}\hookrightarrow\mathfrak{h}^*$ (see 1.2).

Theorems A and B are shown by the Weyl-Kac character formula and its application derived by Jimbo-Miwa [4].

§ 3. **Dualities of branching rules.** We derive two affine cases here. First, noting $\wedge(\hat{W}^-_{4(l+m)n}) \simeq \wedge(\hat{W}^-_{4ln})\otimes\wedge(\hat{W}^-_{4mn})$ we deduce

Corollary 3. *Define the coset Virasoro module* $\mathcal{A}^Y_{y,y'}$ (*resp.* $\mathcal{B}^{y,y'}_Y$) *by*

$$L^{\mathfrak{sp}^\wedge(2l+2m)}(Y,n)\simeq\oplus_{y\in \mathrm{R}_{ln},\,y'\in\mathrm{R}_{mn}}\mathcal{A}^{Y}_{y,y'}\otimes L^{\mathfrak{sp}^\wedge(2l)}(y,n)\otimes L^{\mathfrak{sp}^\wedge(2m)}(y',n)$$

$$(\textit{resp.}\ L^{\mathfrak{sp}^\wedge(2n)}(y,l)\otimes L^{\mathfrak{sp}^\wedge(2n)}(y',m)\simeq\oplus_{Y\in\mathrm{R}_{l+m,n}}\mathcal{B}^{y,y'}_{Y}\otimes L^{\mathfrak{sp}^\wedge(2n)}(y,l+m)).$$

Then $\mathcal{A}^{Y}_{y,y'}\simeq\mathcal{B}^{y\dagger,y'\dagger}_{Y\dagger}$ *for* $y\in\mathrm{R}_{l,n}$, $y'\in\mathrm{R}_{m,n}$ *and* $Y\in\mathrm{R}_{l+m,n}$.

Next we consider the restriction to the subalgebra

$$\mathfrak{sp}(2n)\otimes \boldsymbol{C}[t^l,t^{-l}]\oplus\boldsymbol{C}c\subset\mathfrak{sp}(2n)\otimes\boldsymbol{C}[t,t^{-1}]\oplus\boldsymbol{C}c=\mathfrak{sp}^\wedge(2n).$$

This time the counterpart is $\mathfrak{sp}^\wedge(2m)\overset{\iota}{\hookrightarrow}\mathfrak{sp}^\wedge(2lm)$, where ι is defined by

Lemma. (i) *The following map is a Lie algebra monomorphism.*

$$\iota:\mathfrak{sp}^\wedge(2m)\ni\begin{bmatrix}E^p{}_q & 0\\ 0 & -E^q{}_p\end{bmatrix}(i+kl)\longmapsto\sum_{j=1}^{i}\begin{bmatrix}E^p{}_q\otimes E^j{}_{l+j-i} & 0\\ 0 & -E^q{}_p\otimes E^{l+j-1}{}_j\end{bmatrix}(k+1)$$
$$+\sum_{j=i+1}^{l}\begin{bmatrix}E^p{}_q\otimes E^j{}_{j-i} & 0\\ 0 & -E^q{}_p\otimes E^{j-i}{}_j\end{bmatrix}(k)\in\mathfrak{sp}^\wedge(2lm)$$

$$\begin{bmatrix}0 & E^p{}_q+E^q{}_p\\ 0 & 0\end{bmatrix}(i+kl)\longmapsto\sum_{j=1}^{i}\begin{bmatrix}0 & E^p{}_q\otimes E^j{}_{i-j+1}+E^q{}_p\otimes E^{i-j+1}{}_j\\ 0 & 0\end{bmatrix}(k+1)$$
$$+\sum_{j=i+1}^{l}\begin{bmatrix}0 & E^p{}_q\otimes E^j{}_{l+i-j+1}+E^q{}_p\otimes E^{l+i-j+1}{}_j\\ 0 & 0\end{bmatrix}(k),$$

$$\begin{bmatrix}0 & 0\\ E^p{}_q+E^q{}_p & 0\end{bmatrix}(i+kl)\longmapsto\sum_{j=1}^{l-i}\begin{bmatrix}0 & 0\\ E^p{}_q\otimes E^j{}_{l-i-j+1}+E^q{}_p\otimes E^{l-i-j+1}{}_j & 0\end{bmatrix}(k)$$
$$+\sum_{j=l+1-i}^{l}\begin{bmatrix}0 & 0\\ E^p{}_q\otimes E^j{}_{2l-i-j+1}+E^q{}_p\otimes E^{2l-i-j+1}{}_j & 0\end{bmatrix}(k+1),$$

$$c^{\mathfrak{sp}^\wedge(2m)}\longmapsto c^{\mathfrak{sp}^\wedge(2lm)}\qquad(1\le i\le l,\ k\in\boldsymbol{Z}\ \textit{and}\ 1\le p,\ q\le m).$$

(ii) *Let* $l=2$. *Then* $\iota(\mathfrak{sp}^\wedge(2m))=\{x\in\mathfrak{sp}^\wedge(4m)\,|\,\sigma(x)=x\}$, *where* σ *is the order* 2 *diagram automorphism of* $\mathrm{C}^{(1)}_{2m}$ [4].

Corollary 4. *Define* A^y_Y (*resp.* $\boldsymbol{B}^Y_y$) *by*

$$L^{\mathfrak{sp}^\wedge(2n)}(y,m)\simeq\oplus_{Y\in\mathrm{R}_{n,lm}}A^y_Y\otimes L^{\mathfrak{sp}^\wedge(2n)}(Y,lm)$$

$$(\textit{resp.}\ L^{\mathfrak{sp}^\wedge(2lm)}(Y,n)\simeq\oplus_{y\in\mathrm{R}_{m,n}}\boldsymbol{B}^Y_y\otimes L^{\mathfrak{sp}^\wedge(2m)}(y,n)).$$

Then $A^{y\dagger}_{Y\dagger}\simeq\boldsymbol{B}^Y_y$ *as a Virasoro module, for* $Y\in\mathrm{R}_{lm,n}$ *and* $y\in\mathrm{R}_{m,n}$.

This is our second duality, which is shown by using "principal picture" on $\wedge(\hat{W}^-_{4mn})$. The case $n=1$ of Cor. 3 appears in [6] and [7], and the case $l=2$ of Cor. 4 appears in [4].

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