12. A Proof of Existence of The Stable Jacobi Tensor

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0. Let M^n be a complete Riemannian manifold without conjugate points and R the curvature tensor of M. Let $\gamma:(-\infty,\infty)\to M$ be a geodesic and let $E_1,\cdots,E_n$ be a parallel orthonormal frame field along γ with $E_n(t)=\dot{\gamma}(t)$. We consider the $(n-1)\times(n-1)$ matrix differential equation

(J) $$D''(t)+R(t)D(t)=0,$$

where $R(t)_{ij}=\langle R(E_i,E_n)E_n,E_j\rangle(t)$ for any $t\in(-\infty,\infty)$. If $D(t)$, $-\infty<t<\infty$, is a solution of (J) and x is a parallel vector field along γ, then $D(t)x(t)$, $-\infty<t<\infty$, is a Jacobi field along γ. The following theorems play important roles in the study of manifolds without conjugate points.

Theorem 1. *Let M be a complete simply connected Riemannian manifold without conjugate points and $\gamma:(-\infty,\infty)\to M$ a geodesic. If $D_s(t)$, $-\infty<t<\infty$, are the solution of* (J) *with $D_s(0)=I$, $D_s(s)=0$ for all $s>0$, then the sequence D_s converges to a Jacobi tensor D along γ with $D(0)=I$,* $\det D(t)\neq 0$ *for any $t\in(-\infty,\infty)$ as $s\to\infty$.*

Theorem 2. *Let M and γ be as above. Then, there is a symmetric matrix field A along γ which satisfies the Ricatti equation, namely $A'(t)+A(t)^2+R(t)=0$ for any $t\in(-\infty,\infty)$.*

The theorems were originally proved by Hopf [5] and Green [4] under a more general setting. The proof was explained by Eberlein [1], Eschenburg-O'Sullivan [2] and Goto [3]. The purpose of the present note is to give a geometrical and visual proof which is simpler to some readers. The different point from their proof is that we prove Theorem 2 before Theorem 1. Theorem 1 is an immediate consequence from Theorem 2.

1. Since M is simply connected, all geodesics $\alpha:(-\infty,\infty)\to M$ are minimizing and M is diffeomorphic to E^n. In particular, all spheres are of class C^∞.

Let γ and D_s be as in Theorem 1. $D_s(t)$, $-\infty<t<\infty$, is obtained by the following way: Let $S(\gamma(0),\gamma(s))$ be the sphere with center $\gamma(s)$ through $\gamma(0)$ and let v be the unit normal vector field on $S(\gamma(0),\gamma(s))$ pointing $\gamma(s)$. We consider a map $\phi:S(\gamma(0),\gamma(s))\times(-\infty,\infty)\to M$ given by $\phi(q,t)=\exp tv(q)$. We denote by ϕ_t the map $q\to\phi(q,t)$. If $c:(-\varepsilon,\varepsilon)\to S(\gamma(0),\gamma(s))$, $c(0)=\gamma(0)$, is a curve, then $\phi\circ(c\times id):(-\varepsilon,\varepsilon)\times(-\infty,\infty)\to M$ is a geodesic variation, and, thus, $\phi_{t*}x$, $-\infty<t<\infty$, is a Jacobi field along γ for any $x\in T_{\gamma(0)}S(\gamma(0),\gamma(s))$. Hence,

$$D_s(t)=\phi_{t*}\circ P_t^{-1}$$

for any $t\in(-\infty,\infty)$, where $P_t:T_{\gamma(0)}M\to T_{\gamma(t)}M$ is the parallel translation along γ.

Since M has no conjugate points, $\det D_s(t)\neq 0$ for any $t<s$. If $A_u(t)$ is the second fundamental form of $S(\gamma(t),\gamma(u))$ at $\gamma(t)$ relative to $-\dot{\gamma}(t)$, then we have

$$A_s(t)=D_s'(t)D_s(t)^{-1},$$

and,

$$(*)\qquad A_s'(t)+A_s(t)^2+R(t)=0,$$

for any $t<s$, because $\phi_t(S(\gamma(0),\gamma(s)))=S(\gamma(t),\gamma(s))$ (see [2] or [6]). This implies that $A_s(t)$, $-\infty<t<s$, is the symmetric solution of matrix Ricatti differential equation

$$\text{(R)}\qquad \frac{dX}{dt}+X^2+R(t)=0$$

with initial condition $X(0)=A_s(0)$.

Let $-\infty<t<s$. Then, from Lemma which will be proved in **2**,

$$A_u(t)\geq A_s(t)\geq A_{u'}(t)$$

for any $u<t<u'<s$, namely, for any perpendicular vector x to $\dot{\gamma}(t)$,

$$(**)\qquad \langle A_u(t)x, x\rangle\geq\langle A_s(t)x, x\rangle\geq\langle A_{u'}(t)x, x\rangle,$$

since $S(\gamma(t),\gamma(s))$ separates $S(\gamma(t),\gamma(u))$ from $S(\gamma(t),\gamma(u'))$ in M. In particular, $A_{-1}(0)\geq A_s(0)\geq A_u(0)$, for any $0<u<s$. Therefore, $\{A_s(0)\}$ converges to a symmetric matrix $A(0)$ as $s\to\infty$.

We want to prove that the sequence $\{A_s\}$ of symmetric matrices along γ converges to a solution of (R) which is defined on $(-\infty,\infty)$. Fix a $T>0$. Let $C(T):=\max\{|\langle A_{T+1}(t)x, x\rangle|, |\langle A_{-(T+1)}(t)x, x\rangle|; -T<t<T, x\perp\dot{\gamma}(t), |x|=1\}$. If $s>T+1$, then we see by $(**)$

$$|\langle A_s(t)x, x\rangle|<C(T)$$

for any $-T<t<T$ and $x\perp\dot{\gamma}(t)$, $|x|=1$. Let $E_1,\cdots,E_n$ be a parallel orthonormal frame field along γ with $\dot{\gamma}(t)=E_n(t)$. Set $A_s(t)E_i(t)=\sum_{j=1}^{n-1}a_s(t)_{ij}E_j(t)$ for $t<s$. Then, we have

$$|a_s(t)_{ij}|<2C(T),$$

for any $-T<t<T$, and $i,j=1,2,\cdots,n-1$, since, for $i\neq j$,

$$|\langle A_s(t)((E_i(t)+E_j(t))/\sqrt{2}),\ (E_i(t)+E_j(t))/\sqrt{2}\rangle|<C(T),$$

namely,

$$|\langle A_s(t)E_i(t), E_j(t)\rangle|-(|\langle A_s(t)E_i(t), E_i(t)\rangle| + |\langle A_s(t)E_j(t), E_j(t)\rangle|)/2<C(T).$$

Therefore, we have by integrating $(*)$

$$|a_s(t')_{ij}-a_s(t)_{ij}|\leq\left|\int_t^{t'}\sum_{k=1}^{n-1}a_s(u)_{ik}a_s(u)_{kj}+R(u)_{ij}du\right|\leq C_0(T)\,|t'-t|,$$

for any $t,t'\in[-T,T]$, where $C_0(T):=4(n-1)C(T)^2+\max\{|R(t)_{ij}|; i,j=1,2,\cdots,n-1, -T<t<T\}$. This implies that the sequence $\{A_s\}_{s>T+1}$ is bounded and equicontinuous on $\{-T,T\}$, and, hence, by the convergence property of $\{A_s(0)\}_{s>T+1}$, converges to a symmetric solution $A(t)$, $-\infty<t<\infty$, of (R). Theorem 2 is proved.

Since D_s are Jacobi tensor fields along γ with $D_s(0)=I$ for all $s>0$ and since $D_s'(0)=A_s(0)$ converges to $A(0)$, we see that D_s and D_s' converges to a Jacobi tensor field D and its derivative D' along γ respectively such that

$D(0)=I$ and $D'(t)=A(t)D(t)$ for any $t\in(-\infty,\infty)$. It remains to prove that $\det D(t)\neq 0$ for any $t\in(-\infty,\infty)$. Suppose for indirect proof that $\det D(t_0)=0$ for some t_0. Then, $\operatorname{Ker} D'(t_0)\cap \operatorname{Ker} D(t_0)\neq 0$. This implies that there is a parallel vector field x along γ such that $D(t)x(t)=0$ for any $t\in(-\infty,\infty)$, contradicting that $D(0)=I$. Theorem 1 is proved.

2. Let N be a hypersurface in a Riemannian manifold M and U a small neighborhood of $p\in N$ in M. Assume that N decomposes U into two components and the signed distance function f from N in U is differentiable.

Lemma. *Let N_1 be a hypersurface in U such that $p\in N_1$ and $f(q)\geq 0$ for any $q\in N_1$. If A and A_1 are the second fundamental forms of N and N_1 with same orientation, then $A\leq A_1$ at p.*

Proof. Let $x\in T_pN=T_pN_1$. Let $\gamma:[0,\varepsilon)\to N$ and $\gamma_1:[0,\varepsilon)\to N_1$ be geodesics with $\dot\gamma(0)=\dot\gamma_1(0)=x$. Then, $(f\circ\gamma)''(0)=0$ and $(f\circ\gamma_1)''(0)\geq 0$. Thus,

$$\begin{aligned}\langle A(x),x\rangle&=\langle -\nabla_{\dot\gamma(0)}\operatorname{grad} f,\dot\gamma(0)\rangle=\langle -\nabla_{\dot\gamma_1(0)}\operatorname{grad} f,\dot\gamma_1(0)\rangle\\&=-x\langle \operatorname{grad} f,\dot\gamma_1\rangle+\langle \operatorname{grad} f(p),\nabla_{\dot\gamma_1}\dot\gamma_1(0)\rangle\leq\langle A_1(x),x\rangle.\end{aligned}$$

The lemma is proved.

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