103. Some Remarks on the Generation of Subfields of Ring Class Fields

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1. Introduction. Let K be an imaginary quadratic field and N a prime number which splits in K as: $(N)=\mathfrak{N}\cdot\mathfrak{N}^{\tau}$. Let $\mathcal{O}$ be an order in K and $L=K(j(\mathcal{O}))$, the corresponding ring class field. Deep results were obtained by Ramachandra, Robert, Shintani, Stark and others concerning the generation of subfields of L by values of analytic functions. In this note we observe that under certain conditions, the Siegel units, which are special values obtained from the Δ-function, generate subfields of L. The motivation comes from the study of Heegner points on the modular curve $X_0(N)$. (See [1].) The details of the proofs will appear elsewhere.

2. Notations. In $\mathcal{O}$, write $(N)=\mathfrak{N}_0\cdot\mathfrak{N}_0^{\tau}$ where $\mathfrak{N}_0=\mathfrak{N}\cap\mathcal{O}$. For any subfield F of L containing K, let $cl(L/F)$ denote the subgroup of Pic $\mathcal{O}$ which corresponds to Gal(L/F) under the isomorphism of class field theory Pic $\mathcal{O}\approx$Gal(L/K). Let h be the class number of K and choose $\alpha\in K$ such that $\mathfrak{N}^h=(\alpha)$. Consider the Siegel unit:

$$\varepsilon=\left(\frac{\alpha^{\tau}}{\alpha}\right)^{12}\cdot\left[\frac{\Delta(\mathfrak{N}_0^{\tau})}{\Delta(\mathfrak{N}_0)}\right]^h$$

and let $[\mathfrak{N}_0]$ denote the class of $\mathfrak{N}_0$ in Pic $\mathcal{O}$.

3. Theorem 1. *Notations being as above, we have:*

(i) *If $[\mathfrak{N}_0]$ is not of order dividing* 2 *modulo $cl(L/F)$, then the extension $F/K(N_{L/F}(\varepsilon))$ is at most quadratic.*

(ii) *If $[\mathfrak{N}_0]$ is not of order dividing* 4 *modulo $cl(L/F)$, then*

$$F=K(N_{L/F}(\varepsilon)).$$

As an application we obtain the following:

Theorem 2. *Let p be an odd prime and $\mathcal{O}_n$ the order of K of conductor p^n. Let:*

$$\varepsilon_n=\left(\frac{\alpha^{\tau}}{\alpha}\right)^{12}\cdot\left[\frac{\Delta(\mathfrak{N}_n^{\tau})}{\Delta(\mathfrak{N}_n)}\right]^h$$

where $\mathfrak{N}_n=\mathfrak{N}\cap\mathcal{O}_n$. And let K_∞ denote the anticyclotomic Z_p-extension of K. Then, for n sufficiently large:

$$K_n=K(N_{L_n/K_n}(\varepsilon_n^s))\,;\ s=1,2,3,\cdots$$

where $L_n=K(j(\mathcal{O}_n))$, the ring class field corresponding to $\mathcal{O}_n$ and K_n is the n-th layer of K_∞.

References

[1] B. Gross: Heegner points on $X_0(N)$. In Rankin, R. A. (ed.), Modular Forms. Halsted Press, New York, pp. 87–106 (1984).

[2] K. Ramachandra: Some applications of Kronecker's limit formulas. Ann. of Math., **80**, 104–148 (1964).