100. *Automorphisms of Bounded Reinhardt Domains*

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Introduction. A domain D in C^n is called a *Reinhardt domain* if, for every $(z_1, \cdots, z_n) \in D$ and every $(\alpha_1, \cdots, \alpha_n) \in (U(1))^n$, we heve $(\alpha_1 z_1, \cdots, \alpha_n z_n) \in D$. Here $U(1)$ denotes the multiplicative group of complex numbers of absolute value 1. We are concerned with the determination of holomorphic automorphisms of bounded Reinhardt domains. When the domain contains the origin, this has been completed by Sunada [3]. On the other hand, for the domains not containing the origin, some special cases have been treated in [1]. The purpose of the present note is to determine holomorphic automorphisms of general bounded Reinhardt domains. The details will be given in [2].

1. Algebraic equivalence of Reinhardt domains. Write $T=(U(1))^n$. If D is a Reinhardt domain in C^n, then the group T acts as a group of holomorphic automorphisms on D by the coordinatewise multiplication

$$(\alpha_1, \cdots, \alpha_n)\cdot(z_1, \cdots, z_n)=(\alpha_1 z_1, \cdots, \alpha_n z_n) \quad \text{for} \quad (\alpha_1, \cdots, \alpha_n) \in T \quad \text{and} \quad (z_1, \cdots, z_n) \in D.$$

The subgroup of $\mathrm{Aut}(D)$, the group of all holomorphic automorphisms of D, induced by T is denoted by $T(D)$. Now, biholomorphic mappings between Reinhardt domains equivariant with respect to the T-actions may be considered as natural isomorphisms in the category of Reinhardt domains. The following result gives a characterization of such mappings.

Proposition. *Let $\varphi: D \to D'$ be a biholomorphic mapping between two Reinhardt domains D and D' in C^n. Then φ is equivariant with respect to the T-actions, or equivalently φ has the property that $\varphi T(D)\varphi^{-1}=T(D')$ if and only if it is of the form*

$$(*) \qquad \varphi: D \ni (z_1, \cdots, z_n) \longmapsto (w_1, \cdots, w_n) \in D', \quad w_i=\alpha_i z_1^{a_{1i}} \cdots z_n^{a_{ni}}, \quad i=1, \cdots, n,$$

where $(a_{ij}) \in GL(n, Z)$ and $\alpha_1, \cdots, \alpha_n$ are non-zero complex numbers.

Definition. Two Reinhardt domains in C^n are said to be algebraically equivalent if there is a biholomorphic mapping between them of the form $(*)$.

For a Reinhardt domain D in C^n, we denote by $\mathrm{Aut}_{alg}(D)$ the group of all holomorphic automorphisms of D of the form $(*)$.

2. Automorphisms of Reinhardt domains. If D is a bounded Reinhardt domain in C^n, then an application of a well-known theorem of H. Cartan yields that $\mathrm{Aut}(D)$ has the structure of a Lie group with respect to the compact-open topology. By [1, Section 4, Corollary to Proposition 2], every holomorphic automorphism of D can be written as the composition of an element of $\mathrm{Aut}_{alg}(D)$ and an element of the identity component $G(D)$ of the Lie group $\mathrm{Aut}(D)$. In view of this fact, our main concern is with the determination of the structure of the group $G(D)$.

We now state our result, which generalizes results of [1] and [3].

Theorem. *To each bounded Reinhardt domain D in C^n, there is associated a Reinhardt domain $\tilde{D}$ in C^n which is algebraically equivalent to D such that, for some block decomposition*

$$z=(z^{(1)},\cdots,z^{(r)},z^{(r+1)},\cdots,z^{(s)},z^{(s+1)},\cdots,z^{(t)})\in C^n$$
$$=C^{n_1}\times\cdots\times C^{n_r}\times C^{n_{r+1}}\times\cdots\times C^{n_s}\times C^{n_{s+1}}\times\cdots\times C^{n_t}$$

of coordinates in C^n, the following hold:

i) *$\tilde{D}_1:=p(\tilde{D})$ coincides with $B_{n_1}\times\cdots\times B_{n_r}\times C^{n_{r+1}}\times\cdots\times C^{n_s}$, where p is the projection $C^n\ni z\mapsto(z^{(1)},\cdots,z^{(s)})\in C^{n_1}\times\cdots\times C^{n_s}$ and B_m denotes the unit ball $\{w=(w_1,\cdots,w_m)\in C^m\,|\,|w|:=(|w_1|^2+\cdots+|w_m|^2)^{1/2}<1\}$ in C^m;*

ii) *$\tilde{D}_2:=\{(z^{(s+1)},\cdots,z^{(t)})\in C^{n_{s+1}}\times\cdots\times C^{n_t}\,|\,(0,\cdots,0,z^{(s+1)},\cdots,z^{(t)})\in\tilde{D}\}$ is a bounded Reinhardt domain in $C^{n_{s+1}}\times\cdots\times C^{n_t}$;*

iii) *$\tilde{D}$ can be written in the form*

$$\tilde{D}=\left\{z\in C^n\,|\,(z^{(1)},\cdots,z^{(s)})\in\tilde{D}_1,\left(\frac{z^{(s+1)}}{\prod_{i=1}^{r}(1-|z^{(i)}|^2)^{p_i^{s+1}/2}\prod_{j=r+1}^{s}\exp(-q_j^{s+1}|z^{(j)}|^2)},\cdots,\right.\right.$$
$$\left.\left.\frac{z^{(t)}}{\prod_{i=1}^{r}(1-|z^{(i)}|^2)^{p_i^{t}/2}\prod_{j=r+1}^{s}\exp(-q_j^{t}|z^{(j)}|^2)}\right)\in\tilde{D}_2\right\},$$

where p_i^k, q_j^k, $i=1,\cdots,r$, $j=r+1,\cdots,s$, $k=s+1,\cdots,t$, are non-negative real constants, and, for each index j, $r+1\leq j\leq s$, there is an index k, $s+1\leq k\leq t$, with $q_j^k>0$, $n_k=1$ and $\tilde{D}\cap\{z^{(k)}=0\}=\varnothing$;

iv) *$G(\tilde{D})$ consists of all transformations of the form*

$$\tilde{D}\ni(z^{(1)},\cdots,z^{(t)})\longmapsto(w^{(1)},\cdots,w^{(t)})\in\tilde{D},$$
$$\begin{cases} w^{(i)}=(A^{(i)}z^{(i)}+b^{(i)})(c^{(i)}z^{(i)}+d^{(i)})^{-1}, & i=1,\cdots,r,\\ w^{(j)}=B^{(j)}z^{(j)}+e^{(j)}, & j=r+1,\cdots,s,\\ w^{(k)}=C^{(k)}\prod_{i=1}^{r}(c^{(i)}z^{(i)}+d^{(i)})^{-p_i^k}\prod_{j=r+1}^{s}\exp[-q_j^k\{2({}^t\bar{e}^{(j)}B^{(j)})z^{(j)}+|e^{(j)}|^2\}]z^{(k)}, & \\ & k=s+1,\cdots,t,\end{cases}$$

where

$$\begin{pmatrix}A^{(i)} & b^{(i)}\\ c^{(i)} & d^{(i)}\end{pmatrix}\in SU(n_i,1),\qquad i=1,\cdots,r.$$
$$B^{(j)}\in U(n_j),\quad e^{(j)}\in C^{n_j},\qquad j=r+1,\cdots,s,$$
$$C^{(k)}\in U(n_k),\qquad k=s+1,\cdots,t,$$

and ${}^t\bar{e}^{(j)}$ denotes the transpose of the complex conjugate $\bar{e}^{(j)}$ of $e^{(j)}$.

References

[1] S. Shimizu: Automorphisms and equivalence of bounded Reinhardt domains not containing the origin (to appear in Tôhoku Math. J.).

[2] ——: Automorphisms of bounded Reinhardt domains (to appear).

[3] T. Sunada: Holomorphic equivalence problem for bounded Reinhardt domains. Math. Ann., **235**, 111–128 (1978).