10. The Dimension Formula of the Space of Cusp Forms of Weight One for $\Gamma_0(p)$

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(Communicated by Shokichi IYANAGA, M. J. A., Feb. 12, 1987)

1. Introduction and the statement of the results. We fix an odd prime number p throughout this paper. Let $\Gamma=\Gamma_0(p)$ and χ be a Dirichlet character modulo p satisfying $\chi(-1)=-1$. We regard χ as a character of $\Gamma_0(p)$ by $\chi(\sigma)=\chi(d)$ $\left(\sigma=\begin{bmatrix}a, b\\ c, d\end{bmatrix}\right)$. The purpose of this note is to offer the dimension formula of $S_1(\Gamma_0(p),\chi)$, using Selberg's trace formula. Coauthors obtained the results independently, but decided to publish them together. Details will be published elsewhere.

Let S be the upper half-plane and $G=SL_2(\boldsymbol{R})$. Put $\tilde{S}=S\times(\boldsymbol{R}/2\pi\boldsymbol{Z})$. G acts on $\tilde{S}$ as in [4]. Let $M(\lambda)$ denote the space of f in $L^2(\Gamma\backslash\tilde{S},\chi)$ such that

$$\tilde{\Delta}f=\lambda f,\ \frac{\partial}{\partial\phi}f=-if\ \left(\tilde{\Delta}=y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{5}{4}\frac{\partial^2}{\partial\phi^2}+y\frac{\partial}{\partial x}\frac{\partial}{\partial\phi}\right).$$

Put $\lambda_0=-3/2$. Let $-3/2>\lambda_1>\lambda_2>\lambda_3>\cdots$ be the set of all discrete spectrums of $\tilde{\Delta}$ in $L^2_0(\Gamma\backslash\tilde{S},\chi)$ such that $M(\lambda_i)\neq\{0\}$. It follows from [3] that $S_1(\Gamma_0(p),\chi)$ is isomorphic to $M(\lambda_0)$. Put $d_i=\dim(M(\lambda_i))$, $\lambda_i=-r_i^2-(3/2)$.

For an integral operator K_δ^* (see below), we can rewrite Selberg's trace formula.

Theorem 1. *For* $\mathrm{Re}(\delta)>0$, *we have*

$$(1)\quad \sum_{i=0}^{\infty}h_\delta(r_i)\cdot d_i=J(Id,\delta)+J(E_2,\delta)+J(E_3,\delta)+J(Hyp,\delta)+J(\infty,\delta)+J(0,\delta)$$

$$J(Id,\delta)=2\pi\ \text{volume}\,(\Gamma\backslash S)=(2/3)\pi^2(p+1),\qquad J(E_2,\delta)=0$$

$$J(E_3,\delta)=\frac{8\pi^2}{3\sqrt{-3}\,\delta}\left\{F\left(1,\frac{\delta}{2},1+\delta\,;\,\omega\right)-F\left(1,\frac{\delta}{2},1+\delta\,;\,\bar{\omega}\right)\right\}\alpha_p\beta_\chi$$

$$J(Hyp,\delta)=2^{\delta+2}\pi B\left(\frac{1}{2},\frac{1+\delta}{2}\right)z(\delta,\chi)$$

$$J(\infty,\delta)=J(0,\delta)=\log(\pi/2p^{3/2})g_\delta(0)+\frac{\Gamma\left(\frac{1+\delta}{2}\right)\Gamma\left(\frac{3+\delta}{2}\right)}{4\Gamma\left(1+\frac{\delta}{2}\right)^2}h_{1+\delta}(0)$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(h_\delta(r)+h_{1+\delta}(r)\,\frac{\Gamma\left(\frac{1+\delta}{2}\right)\Gamma\left(\frac{3+\delta}{2}\right)}{\Gamma\left(1+\frac{\delta}{2}\right)^2}\right)\psi(1+ir)dr$$

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$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} h_\delta(r)\left(\frac{L'(1+2ir,\chi)}{L(1+2ir,\chi)}+\frac{L'(1+2ir,\bar{\chi})}{L(1+2ir,\bar{\chi})}\right)dr.$$

Here, $J(*,\delta)$'s denote the contributions from identity, elliptic points of order 4, of order 3 or 6, hyperbolic conjugate classes and cusp at $\infty, 0$. The notations are given as follows; $\alpha_p=1$, $1/2$ or 0 according to $p\equiv 1 \bmod 3$, $p=3$ or $p\equiv 2 \bmod 3$, $\beta_\chi=-2$ or 1 according to $\chi(\sigma)\in \boldsymbol{R}$ or not ($\sigma\in E_3$), $\omega=(1+\sqrt{-3})/2$.

$$(2)\qquad z(\delta,\chi)=\sum_{\{\sigma\}}\log(N(\sigma))\sum_{m=1}^{\infty}\frac{\operatorname{sign}(\lambda(\sigma^m))\chi(\sigma^m)}{(N(\sigma^m)^{1/2}-N(\sigma^m)^{-1/2})(N(\sigma^m)^{1/2}+N(\sigma^m)^{-1/2})^{\delta}}$$

where $\{\sigma\}$ runs over all primitive hyperbolic conjugate classes, and $N(\sigma)$, $\operatorname{sign}(\lambda(\sigma))$ denote the norm of σ, the signature of eigenvalues of σ (Selberg's type zeta function).

$$(3)\qquad \begin{aligned} g_\delta(u)&=2^{\delta+2}\pi B\left(\frac{1}{2},\frac{1+\delta}{2}\right)(e^{u/2}+e^{-u/2})^{-\delta},\\ h_\delta(r)&=2^{\delta+2}\pi B\left(\frac{1}{2},\frac{1+\delta}{2}\right)B\left(\frac{\delta}{2}+ir,\frac{\delta}{2}-ir\right),\end{aligned}$$

ψ denotes the digamma function.

As all terms in (1) except $J(Hyp,\delta)$ can be continued meromorphically in the whole δ-plane, $z(\delta,\chi)$ will be also continued meromorphically there. Since $J(*,\delta)$'s except $J(Hyp,\delta)$ are regular at $\delta=0$, next theorem follows from $\operatorname{Res}_{\delta=0} h_\delta(0)=16\pi^2$.

Theorem 2.

$$(4)\qquad d_0=\frac{1}{4}\operatorname*{Res}_{\delta=0} z(\delta,\chi).$$

2. The Eisenstein series. There are two Γ-inequivalent cusps which are represented by ∞ and 0. For a complex variable t with $\operatorname{Re}(t)>1$, the Eisenstein series are defined by

$$(5)\qquad \begin{aligned} L(2t,\bar{\chi})E_\infty^*(z,\phi,\chi,t)&=\frac{1}{2}\sum_{\substack{(m,n)\in \boldsymbol{Z}\times\boldsymbol{Z},\\ n\not\equiv 0 \pmod p}}{}'\frac{\bar{\chi}(n)y^t e^{-i(\phi+\arg(mpz+n))}}{|mpz+n|^{2t}}\\ L(2t,\chi)E_0^*(z,\phi,\chi,t)&=\frac{-1}{2}\sum_{(m,n)\in\boldsymbol{Z}\times\boldsymbol{Z}}{}'\frac{\chi(m)y^t e^{-i(\phi+\arg(mz+n))}}{p^t|mz+n|^{2t}}.\end{aligned}$$

Lemma 1. *The matrix of constant terms of the Fourier expansion of the Eisenstein series is given in the form*

$$(6)\qquad M(t,\chi)=(m_{\kappa\mu}(t,\chi))=ip^{-t}B\left(t,\frac{1}{2}\right)\begin{bmatrix} 0, & -\dfrac{L(2t-1,\bar{\chi})}{L(2t,\bar{\chi})}\\ \dfrac{L(2t-1,\chi)}{L(2t,\chi)}, & 0\end{bmatrix}.$$

The functional equation of $L(t,\chi)$ gives $M(t,\chi)M(1-t,\chi)=I$.

3. Selberg's trace formula. Now we introduce a point pair G-invariant kernel of (a)–(b) type in the sense of [6]. For $\operatorname{Re}(\delta)>1$, put

$$(7)\qquad \omega_\delta(z,\phi;z',\phi')=\exp(-i(\phi-\phi'))\frac{(yy')^{\delta/2}}{|(z-\bar{z}')/2i|^{\delta}}\frac{(yy')^{1/2}}{(z-\bar{z}')/2i}.$$

It follows from [6] that every non zero element in $M(\lambda)$ is an eigenfunction of the integral operator ω_δ and that its eigenvalue depends only on the spectrum λ; so we use $h_\delta(r)$ for it, which is given in (3). By the aid of the Eisenstein series, we define

$$
\begin{aligned}
&K_\delta^*(z,\phi;z',\phi')=K_\delta(z,\phi;z',\phi')-H_\delta(z,\phi;z',\phi')\\
&K_\delta(z,\phi;z',\phi')=\sum_{\sigma\in\Gamma}\omega_\delta(z,\phi;\sigma(z',\phi'))\chi(\sigma)\\
&H_\delta(z,\phi;z',\phi')=\sum_{\kappa\in\{\infty,0\}}\frac{1}{4\pi^2}\int_{-\infty}^{\infty}h_\delta(r)E_\kappa^*(z,\phi,\chi,t)\overline{E_\kappa^*(z',\phi',\chi,t)}dr\\
&\qquad (t=(1/2)+ir).
\end{aligned}
\tag{8}
$$

By the same argument as in [4, § 4], $K_\delta^*(z,\phi;z',\phi')$ is bounded in $\tilde{S}\times\tilde{S}$, and the integral operator K_δ^* is completely continuous and is zero operator on $L_c^2(\Gamma\backslash\tilde{S},\chi)$. Since we can regard K_δ^* as an operator on $L_0^2(\Gamma\backslash\tilde{S},\chi)$, we get

$$\sum_{i=0}^{\infty}h_\delta(r_i)d_i=\int_{\Gamma\backslash\tilde{S}}K_\delta^*(z,\phi;z,\phi)dzd\phi. \tag{9}$$

Note that the left side will be continued as a meromorphic function of the whole δ-plane with poles at $\{\pm ir_j-2k;\ k\in Z\geqq 0,\ j=0,1,\cdots\}$.

4. The integral over diagonal. In order to calculate the integral in (9), we decompose the integral into the sum of Γ-conjugate classes. Now we assume $\mathrm{Re}(\delta)>1$. Put

$$J(\sigma,\delta)=2\pi\int_{\Gamma(\sigma)\backslash S}\omega_\delta(z,0;\sigma(z,0))dz\chi(\sigma). \tag{10}$$

We can easily verify the contributions from the identity and hyperbolic conjugate classes, so we check only elliptic and cusp's contributions. $J(E_2,\delta)$, $J(E_3,\delta)$ are obtained by the next lemma.

Lemma 2. *For an elliptic element σ, we have*

$$J(\sigma,\delta)=\frac{1}{\delta}\frac{8\pi^2}{(\Gamma(\sigma):\{\pm I\})}\frac{1}{\zeta-\bar{\zeta}}F\left(1,\frac{\delta}{2},1+\delta;1+\zeta^2\right)\chi(\sigma), \tag{11}$$

(F,ζ *being the hypergeometric function, and an eigenvalue of σ chosen as in* [4, § 1]).

The divergence part of the sum of $J(\sigma,\delta)$ over Γ_∞ and Γ_0 is just canceled by that of H_δ ([3]). Take $Y\gg 0$, $D_Y=\{z;\ 0\leqq x\leqq 1,\ 0\leqq y\leqq Y\}$ and $\sigma_\kappa\in G$ such that $\sigma_\kappa(\infty)=\kappa$, $\sigma_\kappa^{-1}\Gamma_\kappa\sigma_\kappa=\Gamma_\infty$. In the same way as in [4], we get

Lemma 3. *The contribution from the cusp at κ is given as*

$$
\begin{aligned}
&2\pi\lim_{Y\to\infty}\left\{\sum_{\substack{\sigma\in\Gamma_\kappa/\{\pm I\},\\ \sigma\neq\{\pm I\}}}\int_{\sigma_\kappa D_Y}\omega_\delta(z,0;\sigma(z,0))dz\right.\\
&\qquad\left.-\int_{\sigma_\kappa D_Y}\frac{1}{4\pi}\int_{-\infty}^{\infty}h_\delta(r)E_\kappa^*(z,0,\chi,t)\overline{E_\kappa^*(z,0,\chi,t)}drdz\right\}\\
&=-\frac{1}{2\pi}\int_{-\infty}^{\infty}h_{1+\delta}(r)\frac{\Gamma\left(\frac{1+\delta}{2}\right)\Gamma\left(\frac{3+\delta}{2}\right)}{\Gamma\left(1+\frac{\delta}{2}\right)^2}\psi(1+ir)dr-\log(2)g_\delta(0)\\
&\quad+\frac{\Gamma\left(\frac{1+\delta}{2}\right)\Gamma\left(\frac{3+\delta}{2}\right)}{4\Gamma\left(1+\frac{\delta}{2}\right)^2}h_{1+\delta}(0)+\frac{1}{4\pi}\int_{-\infty}^{\infty}h_\delta(r)m_{\kappa\mu}(t,\chi)\overline{m'_{\kappa\mu}(t,\chi)}dr\\
&\qquad(t=(1/2)+ir,\ \mu\neq\kappa).
\end{aligned}
\tag{12}
$$

The last term of (12) is transformed into

$$\log(\pi/p^{2/3})g_\delta(0)-\frac{1}{2\pi}\int_{-\infty}^{\infty}h_\delta(r)\psi(1+ir)dr,$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}h_\delta(r)\left(\frac{L'(2t,\chi)}{L(2t,\chi)}+\frac{L'(2t,\bar{\chi})}{L(2t,\bar{\chi})}\right)dr.$$

Since each term in (11), (12) is a holomorphic function in $\mathrm{Re}(\delta)>0$, we get Theorem 1.

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