85. *Reduced Group C*-Algebras with the Metric Approximation Property by Positive Maps*

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1. Introduction. Choi and Effros [3] and Kirschberg [6] have proved that the nuclearity for a C^*-algebra is equivalent to "the complete positive approximation property". Not all C^*-algebras have the approximation property. In fact, A. Szankowski [8] has proved, that the algebra of all bounded operators $B(H)$ on an infinite dimensional Hilbert space H, does not have the approximation property. It had been believed that every C^*-algebra with the metric approximation property is nuclear. Surprisingly, in 1979, Uffe Haagerup [5] showed an example of a non-nuclear C^*-algebra, which has the metric approximation property. Haagerup's example is the reduced group C^*-algebra $C_r^*(F_2)$ of the free group on two generators F_2. In the sequel, Canniere and Haagerup [1] showed that for any fixed $n \in N$, the identity map of $C_r^*(F_2)$ can be approximated by n-positive finite rank operators on $C_r^*(F_2)$. In this note, we shall show that the identity map of the reduced group C^*-algebras generated by the free product of finite groups with one amalgamated subgroup can be approximated by n-positive maps as well. This is an improvement of our previous result in [4].

2. Results. Let $G=A \underset{C}{*} B$ be the free product of two finite groups A and B with one amalgamated subgroup C (cf. [7]). Then there is a tree X on which G acts as follows: Put

$V(X)=(G/A)\cup(G/B)$(disjoint union), the set of vertices of X.

$E(X)=(G/C)\cup\overline{(G/C)}$(disjoint union), the set of edges of X.

The source map $s: G/C \to G/A$ and the range map $r: G/C \to G/B$ are induced by the inclusions $C \to A$ and $C \to B$. An action of G on the tree X is given by $g\cdot(xA)=(gx)A \in V(X)$, $g\cdot(xB)=(gx)B \in V(X)$ and $g\cdot(xC)=(gx)C \in E(X)$ for all g, x in G. Put $P_0=A \in V(X)$. For g in G, define $\Psi(g)=d(P_0, gP_0)$ to be the distance from P_0 to gP_0. Then Ψ is a length function on G[2], [9] such that $\Psi(g)$ is an even integer for all $g \in G$. Note that edges of X consist of

$$xA \circ\!\xrightarrow{\{xC,\ \overline{xC}\}}\!\circ\, xB \qquad x \in G$$

If $d(P_0, Q)$ is even (resp. odd) for $Q \in V(X)$, then $Q=gA=gP_0$ (resp. $Q=gB$) for some $g \in G$. For s in G and integers k, $l \geqq 0$, put

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$$Y(s;k,l)=\{(t,u)\in G\times G;\ s=tu,\ \Psi(t)=k \text{ and } \Psi(u)=l\}.$$

The cardinality of a set S is denoted by $\sharp S$. We need the following lemmas.

Lemma 1. *For $s\in G$, integers k, $l\geqq 0$ with $\Psi(s)=k+l$ or $\Psi(s)=k+l-2$, we have that $\sharp Y(s;k,l)\leqq(\sharp A)\cdot(\sharp B)$.*

We shall show a lemma about a decomposition of an element in G.

Lemma 2. *Suppose that $s=tu$ and $\Psi(s)=\Psi(t)+\Psi(u)-2p$ for $s,t,u\in G$ and an integer $p\geqq 0$. If p is even, then there exist $t',u',v\in G$ such that $t=t'v$, $u=v^{-1}u'$, $\Psi(t')=\Psi(t)-p$, $\Psi(u')=\Psi(u)-p$ and $\Psi(v)=p$. If p is odd, then there exist $t',u',v\in G$ such that $t=t'v$, $u=v^{-1}u'$, $\Psi(t')=\Psi(t)-p-1$, $\Psi(u')=\Psi(u)-p+1$ and $\Psi(v)=p+1$.*

Put $E_n=\{s\in G;\ \Psi(s)=n\}$ and χ_n the characteristic function for E_n.

Lemma 3. *Let k, l, m be non-negative integers and f, g be two functions on G with support in E_k and E_l respectively. Then*

$$\|(f*g)\chi_m\|_2\leqq\{(\sharp A)\cdot(\sharp B)\}^{3/2}\|f\|_2\cdot\|g\|_2 \text{ if } |k-l|\leqq m\leqq k+l$$
$$\text{and } k+l-m \text{ even and } \|(f*g)\chi_m\|_2=0 \text{ if not.}$$

Let λ be the left regular representation of G. For a function $f\in l^1(G)$ we put as usual $\lambda(f)=\sum_{s\in G}f(s)\lambda(s)$.

Lemma 4. *Let f be a function on G, with finite support, then $\|\lambda(f)\|\leqq 2(\sharp A\cdot\sharp B)^{3/2}(\sum_{s\in G}|f(s)|^2(1+\Psi(s))^4)^{1/2}$.*

Let $A(G)$ be the Fourier algebra of G.

Lemma 5. *Let $G=A\underset{C}{*}B$ and $n\in N$. Let Φ be a function on G. If $\Phi(s^{-1})=\overline{\Phi(s)}$ for $s\in G$, $(\sharp A\cdot\sharp B)^3\cdot|\Phi(s)|\cdot(1+\Psi(s))^4\leqq(1/n)\Phi(e)$ for $s\in G\backslash C$ and $\Phi(s)=0$ for $s\in C\backslash\{e\}$, then Φ is an n-positive multiplier on $A(G)$.*

We are now able to prove our main theorem.

Theorem 6. *Let $G=A\underset{C}{*}B$ be the free product of finite groups A and B with one amalgamated subgroup C. There exists a sequence $(\xi_k)_{k\in N}$ of functions with finite support, such that*

(1) *Each ξ_k is a n-positive multiplier of $A(G)$ and $\xi_k(e)=1$.*

(2) $\lim_{k\to\infty}\|\xi_k\Phi-\Phi\|_{A(G)}=0$ *for any $\Phi\in A(G)$.*

Corollary 7. *Let G be as Theorem 6 and $n\in N$. There exists a sequence $(T_k)_{k\in N}$ of n-positive linear maps on the reduced group C^*-algebra $C_r^*(G)$ of G such that (1) Each T_k is of finite rank, and $T_k(1)=1$ (2) $\lim_{k\to\infty}\|T_kx-x\|=0$ for any x in $C_r^*(G)$.*

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