83. *On a Codimension 2 Bifurcation of Heteroclinic Orbits*

By Hiroshi KOKUBU
Department of Mathematics, Kyoto University

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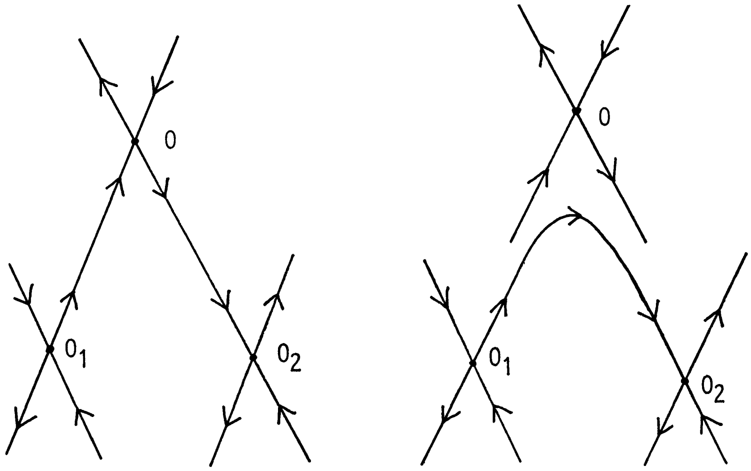
1. **Introduction.** We consider a bifurcation problem of heteroclinic orbits for a family of ODEs on $\boldsymbol{R}^n$. Suppose there are two heteroclinic orbits, one of which connects saddle points O_1 and O, the origin, and the other connects O and O_2. See Figure (a) below.



(a) (b)

Figure

In general, these heteroclinic orbits are broken by perturbations, since they are structurally unstable. We will give below, under some non-degeneracy assumptions, a condition of parameter values for which each heteroclinic orbit persists, and also a condition for which there is a new heteroclinic orbit (Figure (b)) connecting O_1 and O_2 given by joining original heteroclinic orbits near the origin O.

Recent developement of the theory of Melnikiov functions and the exponential dichotomy [3] invoked many works on bifurcations of homoclinic (heteroclinic) orbits, most of which are related to the codimension 2 bifurcation of the vector field singularities ([2], [4] and references therein). From a bifurcation theoretical point of view, it seems more difficult to treat bifurcations of homoclinic and heteroclinic orbits than those of equilibria or periodic orbits, since the formers are global ones.

2. Assume a smooth ODE family $\dot{x}=f(x)+g(x,\mu)$ $(x\in\boldsymbol{R}^n,\ \mu\in\boldsymbol{R}^k)$ with $f(0)=g(0,\mu)=g(x,0)=0$, has three saddle points, $O_1(\mu)$, $O_2(\mu)$ and the origin O. The eigenvalues of the Jacobian matrix at each equilibrium O [resp.

$O_i(\mu)$] are assumed to be $\nu(\mu)$, $-\rho(\mu)$ and $-\eta^k(\mu)$ [resp. $\nu_i(\mu)$, $-\rho_i(\mu)$ and $-\eta_i^k(\mu)$] ($i=1,2,1\leq k\leq n-2$) satisfying

$$\nu(\mu)>0>-\rho(\mu)>-\mathcal{R}e(\eta^k(\mu)) \text{ [resp. } \nu_i(\mu)>0>-\rho_i(\mu)>-\mathcal{R}e(\eta_i^k(\mu))].$$

At $\mu=0$, we suppose that there exist two heteroclinic orbits $h_i(t)$ ($i=1,2$) connecting $O_1(\mu)$ and O for $i=1$, and O and $O_2(u)$ for $i=2$. Then we can show that the linear ODE $\dot{z}=Df(h_i(t))\cdot z$ has the exponential dichotomy on both intervals $\boldsymbol{R}_-=(-\infty,0]$ and $\boldsymbol{R}_+=[0,+\infty)$, that is, the fundamental solution matrix $X_i(t)$ satisfies

$$|X_i(t)\cdot P_\pm\cdot X_i^{-1}(s)|\leq Ke^{-\alpha(t-s)} \qquad \text{for any } s,\ t\in \boldsymbol{R}_\pm \text{ with } s\leq t,$$

and

$$|X_i(t)\cdot(I-P_\pm)\cdot X_i^{-1}(s)|\leq Ke^{-\alpha(s-t)} \qquad \text{for any } s,\ t\in \boldsymbol{R}_\pm \text{ with } s\geq t,$$

where K and α are positive constants, and $P_\pm$ are projection matrices.

Further, we make the following assumptions:

(G1) Heteroclinic orbits are generic in the sense that, as $t\to+\infty$, each of them approaches to an equilibrium along the eigenspace associated with $-\rho(\mu)$ [resp. $-\rho_2(\mu)$].

(G2) $$\int_{-\infty}^{+\infty}\hat{q}^i(s)\cdot\frac{\partial}{\partial\mu}g(h_i(s),0)ds\neq 0, \qquad (i=1,2),$$

and these integral vectors are linearly independent, where $\hat{q}^i(s)$ is a bounded solution of the linear ODE $\dot{\hat{z}}=-{}^tDf(h_i(t))\cdot\hat{z}$.

(G3) For μ sufficiently small and all j, k with $1\leq j\leq k\leq n-2$,

$$\nu(\mu)-\mathcal{R}e(\eta^k(\mu))\neq-\rho(\mu),\ -\mathcal{R}e(\eta^j(\mu)).$$

Under the above hypotheses except (G3), we can prove the following theorem using the standard theory on the exponential dichotomy (cf. [2], [5]).

Theorem 1. *In a neighborhood of $\mu=0$, there exist two codimension 1 hypersurfaces M_i ($i=1,2$) intersecting transversally at $\mu=0$, in which each μ corresponds to a parameter value having a heteroclinic orbit connecting O and $O_i(\mu)$.*

Moreover, under (G3) as well as (G1) and (G2), we can prove:

Theorem 2. *There exists a codimension 1 hypersurface M_{12} containing $\mu=0$ at its boundary, in which each μ corresponds to a parameter value having a heteroclinic orbit connecting $O_1(\mu)$ and $O_2(\mu)$. Furthermore,*

(a) *if $\nu(0)<\rho(0)$ then M_{12} is tangent to M_2 at $\mu=0$.*

(b) *if $\nu(0)=\rho(0)$ and $d/d\mu|_{\mu=0}\{\nu(\mu)-\rho(\mu)\}\neq 0$ then M_{12} is tangent to neither of M_i.*

(c) *if $\nu(0)>\rho(0)$ then M_{12} is tangent to M_1 at $\mu=0$.*

Remark. (1) The assumptions (G1)–(G3) are open conditions, hence our theorems show a generic codimension 2 bifurcation of such heteroclinic orbits.

(2) The bounded solution $\hat{q}^i(t)$ is unique up to multiplication by a constant.

(3) Without losing generality, we can assume that $\mu=(c,\lambda)\in\boldsymbol{R}\times\boldsymbol{R}^{k-1}$ and that each M_i [resp, M_{12}] is given as the graph of $c=c_i(\lambda)$ [resp.

$c=c(\lambda)$] near $\lambda=0$. In what follows we use such (c, λ).

(4) The conclusions of Theorems 1 and 2 are also valid for the case that some of the saddle points coinside. Such cases correspond to those of homoclinic orbits. For example, in case $O_1=O_2$, we can show that there are two homoclinic orbits with saddle points O and O_1 $(=O_2)$ respectively, besides the original heteroclinic orbits, and that the bifurcation set of each homoclinic orbit is tangent to one of the bifurcation set of heteroclinic orbits at $\mu=0$, if the first two eigenvalues ν and $-\rho$ of the saddle point of the homoclinic orbit satisfies $\nu\neq\rho$.

3. In order to prove Theorem 2, we need a lemma describing the behavior of orbits near the saddle point O. Let Σ_s [resp. Σ_u] be a plane at the distance of sufficiently small $\delta>0$ from O, and transverse to the heteroclinic orbit $h_1(t)$ [resp. $h_2(t)$] when $\mu=0$. For small $\mu\neq 0$, $W^u(O_1(\mu))\cap\Sigma_s$ defines a point $x(\mu)$. Set $\alpha(\mu)$ be the distance of $x(\mu)$ and $W^s(O)$. Similarly, let the point $x'(\mu)$ be the intersection of the orbit starting $x(\mu)$ and Σ_u, and define $\alpha'(\mu)$ be the distance of $x'(\mu)$ and $W^u(O)$.

Lemma. *Under the above notations, it holds that*

$$\alpha'(\mu)=A(\mu)\cdot\alpha(\mu)^{\rho(\mu)/\nu(\mu)}$$

such that, recalling $\mu=(c, \lambda)$,

$$A(0)\neq 0 \quad \textit{and} \quad \lim_{c\to c_1(\lambda)}\frac{\partial A}{\partial\mu}\cdot\alpha^{\rho/\nu}=0 \quad \textit{for} \quad \frac{\rho(c_1(\lambda), \lambda)}{\nu(c_1(\lambda), \lambda)}\geq 1.$$

This lemma can be proved using the C^1-linearization theorem by Belitskii [1] under (G3). Our Theorem 2 is obtained by this lemma with the aid of the exponential dichotomy technique.

4. It is often the case that the existence of a heteroclinic (homoclinic) orbit automatically implies the existence of another heteroclinic (homoclinic) orbit; for instance, under the presence of a symmetry. The method to prove Theorems 1 and 2 can be used for such cases. Especially, we have the following theorem assuming that $x\in\boldsymbol{R}^3$ and $\mu\in\boldsymbol{R}^2$ for simplicity.

Theorem 3. *Suppose there is a homoclinic orbit of the origin O at $\mu=0$, then it persists on a curve M_0 passing through the origin given as a graph of a function $c=c_0(\lambda)$ in a neighborhood of O in the parameter space. Moreover, if $\nu(0)=\rho(0)$, $d/d\lambda|_{\lambda=0}\{\nu(c_0(\lambda), \lambda)-\rho(c_0(\lambda), \lambda)\}\neq 0$, and $A(0)\neq 1$, then there exists a curve M branching off from $\mu=0$ tangentially to M_0, which consists of parameter values corresponding to homoclinic orbits rounding twice along the original homoclinic orbit.*

Remark. (1) This theorem was obtained by Yanagida [7] but his proof is insufficient at the point that he used the C^0-linearization, hence his bifurcation analysis is not rigorous as it is.

(2) As for the bifurcation of homoclinic orbits, there is a remarkable theorem by Sil'nikov [6] which shows the existence of the chaotic dynamics near a homoclinic orbit of the saddle-focus type. On the other hand, Theorem 3 gives a bifurcation of a homoclinic orbit of the saddle-node type.

We can obtain similar theorems for various kinds of homoclinic and heteroclinic orbits under the Z_2-symmetry.

The details of the results in this paper as well as proofs will appear elsewhere.

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