80. *A Generalization of Itô's Lemma*

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1. Introduction. In the traditional definition of K. Itô's stochastic integral of a process φ with respect to Brownian motion B it is essential that φ be non-anticipatory [8]. However, there are some works in which one has tried to avoid this condition, s. e.g. [1, 4, 9]. Finally, the white noise analysis, advocated by T. Hida (e.g. [2, 3]), has provided a framework, in which stochastic integrals can be naturally defined without posing such measurability conditions, as has been shown in a recent paper by H.-H. Kuo and A. Russek [7].

Let $(\mathcal{S}'(\boldsymbol{R}), \mathcal{B}, d\mu)$ be white noise, i.e. $\mathcal{B}$ is the σ-algebra over $\mathcal{S}'(\boldsymbol{R})$ generated by the cylinder sets and μ is the Gaussian measure on $\mathcal{B}$ with characteristic functional

(1.1) $$\exp(-1/2\,\|\xi\|_2^2)=\int_{\mathcal{S}'(\boldsymbol{R})}\exp(i\langle x,\xi\rangle)d\mu(x)$$

for $\xi\in\mathcal{S}(\boldsymbol{R})$, $\|\cdot\|_2$ denoting the norm of $L^2(\boldsymbol{R}, dt)$ and $\langle\cdot,\cdot\rangle$ the canonical duality. By (L^p), $p>0$, we denote the Banach space $L^p(\mathcal{S}'(\boldsymbol{R}), \mathcal{B}, d\mu)$. Note that

(1.2) $$B(t\,;x):=\langle x, 1_{(0,t)}\rangle, \qquad x\in\mathcal{S}'(\boldsymbol{R})$$

(although not pointwise defined) is a well-defined random variable in (L^p), $p\geq 1$, and a Brownian motion (under $d\mu$).

In [2, 3] Hida introduced the space $(L^2)^+$ of testfunctionals of white noise and its dual $(L^2)^-$ of generalized functionals. Furthermore he defined the operators ∂_t, $t\in\boldsymbol{R}$, which are partial derivatives $\partial/\partial x(t)$ for white noise testfunctionals, cf. also [5, 6]. Since ∂_t is densely defined on $(L^2)^+$ there is its adjoint ∂_t^* acting on $(L^2)^-$. Note that we have the Gel'fand triple

(1.3) $$(L^2)^-\supset(L^2)\supset(L^2)^+$$

so that ∂_t^* acts by restriction on (L^2).

The following was shown in the paper [7] of Kuo and Russek: assume that φ is a map from $\boldsymbol{R}_+$ into (L^2), non-anticipatory (i.e. for each $t\in\boldsymbol{R}_+$, $\varphi(t)$ is measurable w.r.t. $\sigma(B(s\,;\cdot), 0\leq s\leq t)$) and

(1.4) $$\int_a^b \boldsymbol{E}(|\varphi(t)|^2)dt$$

is finite, then

(1.5) $$\int_a^b \partial_t^*\varphi(t)dt$$

exists in (L^2) and equals Itô's stochastic integral of φ w.r.t. Brownian motion. Of course, this generalizes to higher-dimensional Brownian motions, using independent copies of white noise.

The important thing about (1.5) is that this expression is defined independently of any measurability conditions posed on $\varphi(t)$ (except for $\mathscr{B}$-measurability). Only (1.4) has to be supplemented with the condition that $\int_{[a,b]^2} E(\partial_s\bar{\varphi}(t)\partial_t\varphi(s))dsdt$ be finite (cf. [7] for the action of ∂_t on (L^2)). Hence (1.5) is an extension of the traditional definition for quite general processes in (L^2).

For the theory and applications of stochastic integrals Itô's lemma [8] plays a key role. Naturally the question arises, what this lemma becomes, if the non-anticipatory condition on φ is dropped and stochastic integrals are understood as in (1.5). The answer is given in the next section.

2. A generalization of Itô's Lemma. Let φ_i, ψ_i, $i=1,2,\cdots,n$ be real, strongly continuous processes in (L^2), such that the stochastic integrals

$$X_i(t):=x_i+\int_0^t \partial_s^*\varphi_i(s)ds+\int_0^t \psi_i(s)ds \tag{2.1}$$

exist in (L^6) for all $t\in \boldsymbol{R}_+$. Here $x_i\in \boldsymbol{R}$, $i=1,2,\cdots,n$.

The discussion after (1.5) implies, that we have to set

$$\int_0^t \eta(s)dX_i(s)=\int_0^t \partial_s^*\eta(s)\varphi_i(s)ds+\int_0^t \eta(s)\psi_i(s)ds \tag{2.2}$$

for the stochastic integral of another process η w.r.t. dX_i: (2.2) guarantees that this stochastic integral coincides with the conventional one, if all processes involved are non-anticipatory w.r.t. Brownian motion.

For the following let $\eta: \boldsymbol{R}_+\to(L^2)$ be continuous and assume that $\eta(t)$ has a piecewise continuous integral kernel (cf. [2,3,6]). Furthermore let (Δ_l) be a partition of $(0,t)$ into intervals $(t_l, {}_{l+1})$ of length Δ and put

$$Y_i(a,b)=\int_a^b \partial_s^*\varphi_i(s)ds,\ \mathrm{i}=1,2,\cdots,n \tag{2.3}$$

Lemma.

a) (2.4)
$$\lim_{\Delta\downarrow 0}\sum_l \eta(t_l)Y_i(t_l,t_{l+1}) =\int_0^t [\partial_s^*\eta(s)\varphi_i(s)+(\partial_{s+}\eta(s))\varphi_i(s)]ds$$
where ∂_{s+} is the derivative defined in [7]

b) (2.5)
$$\lim_{\Delta\downarrow 0}\sum_l \eta(t_l)Y_i(t_l,t_{l+1})Y_j(t_l,t_{l+1}) =\int_0^t \eta(s)\varphi_i(s)\varphi_j(s)ds$$

c) (2.6)
$$\lim_{\Delta\downarrow 0}\sum_l \eta(t_l)Y_i(t_l,t_{l+1})Y_j(t_l,t_{l+1})Y_k(t_l,t_{l+1}) =0$$

(all limits are taken in the topology of (L^2)).

The proof of this lemma is performed by straightforward computations of the $\mathcal{S}$-transforms [3,5,6] of products of random variables and standard estimations of the resulting Lebesgue integrals.

Now let $\underline{X}(t)$ denote the $\boldsymbol{R}^n$-valued random variable with components

$X_i(t)$, $i=1,2,\cdots,n$, and consider its composite $F\circ \underline{X}(t)\equiv F(\underline{X}(t))$ with a real $C^3(\boldsymbol{R}^n)$ function F. Furthermore we have to assume that F and its partial derivatives $D^\alpha F$, $|\alpha|=1,2,3$, composed with $\underline{X}\,(t)$ belong to (L^2) for all t.

Under the preceding conditions it is a matter of applying Taylor's theorem and the lemma to establish the following theorem:

Theorem.

$$
\begin{aligned}
&F(\underline{X}(t))-F(\underline{X}(s))\\
&\quad=\int_s^t \sum_{i=1}^n D^i F(\underline{X}(u))dX_i(u)\\
&\quad+\int_s^t \sum_{i,j=1}^n D^{ij}F(\underline{X}(u))[1/2\varphi_i(u)\varphi_j(u)+\varphi_i(u)\partial_{u+}X_j(u)]du
\end{aligned}
\tag{2.7}
$$

Note that formula (2.7) reduces to the usual Itô-lemma, in case that φ_i and ψ_i are non-anticipatory for all $i\in\{1,\cdots,n\}$, since then also all the X_i are non-anticipating and $\partial_{u+}X_i(u)=0$ by a theorem in [7].

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