9. *The Euler Number and Other Arithmetical Invariants for Finite Galois Extensions of Algebraic Number Fields*

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1. Introduction. Let k be an algebraic number field of finite degree over the rational field $\boldsymbol{Q}$. Recently, T. Ono introduced new arithmetical invariants $E(K/k)$ and $E'(K/k)$ for a finite extension K/k. In [5], he obtained a formula between the Euler number $E(K, k)$ and other cohomological invariants for a finite Galois extension K/k. In [2], we obtained a similar formula for $E'(K/k)$. Both proofs use Ono's results on the Tamagawa number of algebraic tori, on which the formulae themselves do not depend. The purpose of this paper is to give a direct proof of these formulae, in response to a problem posed by T. Ono [6]. At the same time, we shall get some relations between $E(K/k)$, $E'(K/k)$ and other arithmetical invariants of K/k (for example, central class number, genus number etc.).

2. Let T be an algebraic torus defined over k, and K- be a Galois splitting field of T. We denote the Galois group $\mathrm{Gal}(K/k)$ by G and the character module $\mathrm{Hom}(T, G_m)$ by $\hat{T}$. $\hat{T}_0$ denotes the integral dual of $\hat{T}$. Let $T(k_A)$, $T(k)$ and $T(k_\mathfrak{p})$ be the k-adelization of T, k-rational points of T and $k_\mathfrak{p}$-rational points of T, where $\mathfrak{p}$ is a place of k. When $\mathfrak{p}$ is finite, we denote the unique maximal compact subgroup of $T(k_\mathfrak{p})$ by $T(O_\mathfrak{p})$. $T(U_k)$ denotes the group

$$\prod_{\mathfrak{p}:\text{finite}} T(O_\mathfrak{p}) \times \prod_{\mathfrak{p}:\text{infinite}} T(k_\mathfrak{p}),$$

where $\mathfrak{p}$ runs over all the places of k. We define the class group of T by putting

$$C(T) = T(k_A)/T(k)\cdot T(U_k).$$

As G-modules, we have

$$T(k_A) \cong (\hat{T}_0 \otimes K_A^\times)^G,$$
$$T(k) \cong (\hat{T}_0 \otimes K^\times)^G,$$
$$T(U_k) \cong (\hat{T}_0 \otimes U_K)^G.$$

Here

$$U_K = \prod_{\mathfrak{P}:\text{finite}} O_\mathfrak{P}^\times \times \prod_{\mathfrak{P}:\text{infinite}} K_\mathfrak{P}^\times,$$

where $\mathfrak{P}$ runs over all the places of K. We note here that $h(T)$, the class number of the torus T, is the order of the group $C(T)$. First, we shall sketch a new direct proof of the equation between $E(K/k)$ and the cohomological invariants of K/k. Consider the following exact sequence of algebraic tori defined over k

$$(1) \qquad 0 \longrightarrow R_{K/k}^{(1)}(G_m) \longrightarrow R_{K/k}(G_m) \xrightarrow{N} G_m \longrightarrow 0.$$

In this section, we shall denote $R^{(1)}_{K/k}(G_m)$, $R_{K/k}(G_m)$, G_m by T', T, T''. Then we have $\hat{T}'_0=\boldsymbol{I}[G]$, $\hat{T}_0=\boldsymbol{Z}[G]$, $\hat{T}''_0=\boldsymbol{Z}$, where $\boldsymbol{I}[G]$ is the augmentation ideal of the group ring $\boldsymbol{Z}[G]$. We denote $\{x \in K_A^\times \mid N_{K/k}(x)=1\}$ by $N^{-1}(1)$. Then, from the cohomology sequence derived from (1), we have

$$T'(k_A)\cong(I[G]\otimes K_A^\times)^G\cong N^{-1}(1),$$
$$T'(k)\cong(I[G]\otimes K^\times)^G\cong N^{-1}(1)\cap K^\times,$$
$$T'(U_k)\cong(I[G]\otimes U_K)^G\cong N^{-1}(1)\cap U_K.$$

Consider a homomorphism $\alpha: C(T')\to C(T)$. Then, from the fact that $C(T)\cong K_A^\times/U_K\cdot K^\times$ and $C(T')\cong N^{-1}(1)/(N^{-1}(1)\cap K^\times)(N^{-1}(1)\cap U_K)$, it is easy to show $\mathrm{Cok}\,\alpha\cong K_A^\times/N^{-1}(1)\cdot U_K\cdot K^\times$. We note here that $\mathrm{Cok}\,\alpha$ is isomorphic to the central class group of K/k. We denote the order $[\mathrm{Cok}\,\alpha]$ by $z_{K/k}$. On the other hand, we have

$$\mathrm{Ker}\,\alpha\cong N^{-1}(1)\cap U_K\cdot K^\times/(N^{-1}(1)\cap U_K)(N^{-1}(1)\cap K^\times)$$
$$\cong O_k^\times\cap N_{K/k}K^\times/N_{K/k}O_K^\times,$$

where $O_k^\times$ and $O_K^\times$ are the unit groups of k and K, respectively. Hence, we have

Theorem 1. *The following sequence is exact*

$$0\longrightarrow O_k^\times\cap N_{K/k}K^\times/N_{K/k}O_K^\times\longrightarrow C(T')\stackrel{\alpha}{\longrightarrow}C(T)\longrightarrow K_A^\times/N^{-1}(1)\cdot U_K\cdot K^\times\longrightarrow 0.$$

Since each group in the sequence of the above theorem is finite, we have the following equation

(2) $$h_K\cdot[O_k^\times\cap N_{K/k}K^\times : N_{K/k}O_K^\times]=h_{K/k}\cdot z_{K/k},$$

where $h_{K/k}$ denotes the order $[C(T')]$. It is easy to prove the following well known result on $z_{K/k}$

$$z_{K/k}=\frac{h_k\cdot i(K/k)\cdot[U_k:N_{K/k}U_K]}{[K_0:k]\cdot[O_k^\times:O_k^\times\cap N_{K/k}K^\times]},$$

where K_0 is the maximal abelian extension of k contained in K, and $i(K/k)=[k^\times\cap N_{K/k}K_A^\times : N_{K/k}K^\times]$. Therefore, from (2), we have

(3) $$E(K/k)=\frac{h_K}{h_k\cdot h_{K/k}}=\frac{i(K/k)\cdot[U_k:N_{K/k}U_K]}{[K_0:k]\cdot[O_k^\times:N_{K/k}O_K^\times]}$$
$$=\frac{z_{K/k}}{h_k\cdot[O_k^\times\cap N_{K/k}K^\times:N_{K/k}O_K^\times]}.$$

Let $g_{K/k}$ be the genus number of K/k, that is the order of the genus group $K_A^\times/N_{K/k}^{-1}(k^\times)\cdot U_K$. From the relation between $g_{K/k}$ and $z_{K/k}$, we have

(4) $$E(K/k)=\frac{i(K/k)\cdot g_{K/k}}{h_k\cdot[O_k^\times\cap N_{K/k}K_A^\times:N_{K/k}O_K^\times]}.$$

3. In this section, we shall consider the following exact sequence of algebraic tori defined over k

(5) $$0\longrightarrow G_m\longrightarrow R_{K/k}(G_m)\longrightarrow R_{K/k}(G_m)/G_m\longrightarrow 0.$$

In the following, we shall denote G_m, $R_{K/k}(G_m)$, $R_{K/k}(G_m)/G_m$ by T', T, T'', respectively. Then we have

$$\hat{T}'_0=\boldsymbol{Z},\quad \hat{T}_0=\boldsymbol{Z}[G],\quad \hat{T}''_0=\boldsymbol{Z}[G]/\boldsymbol{Z}s\cong\boldsymbol{Z}[G]/\boldsymbol{Z},$$

where $s=\sum_{\sigma\in G}\sigma$. Let us consider a homomorphism $\beta: C(T)\to C(T'')$. From Hilbert Theorem 90, we see the homomorphism $T(k_A)\to T''(k_A)$ is surjective.

Since $T''(U_k)\cong((\boldsymbol{Z}[G]/\boldsymbol{Z})\otimes U_K)^G$, $T''(k)\cong((\boldsymbol{Z}[G]/\boldsymbol{Z})\otimes K^\times)^G$, we have the following commutative diagram with exact rows and columns

$$
(6)\qquad
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & U_k & \longrightarrow & U_K & \xrightarrow{g} & ((\boldsymbol{Z}[G]/\boldsymbol{Z})\otimes U_K)^G & \longrightarrow & H^1(G, U_K) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & k_A^\times & \longrightarrow & K_A^\times & \xrightarrow{g} & ((\boldsymbol{Z}[G]/\boldsymbol{Z})\otimes K_A^\times)^G & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & I_K^G & \longrightarrow & I_K & \xrightarrow{\bar{g}} & ((\boldsymbol{Z}[G]/\boldsymbol{Z})\otimes I_K)^G & \longrightarrow & 0 & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}
$$

where I_K is the ideal group of K.

Since $g(K^\times)=((\boldsymbol{Z}[G]/\boldsymbol{Z})\otimes K^\times)^G$, we have

$$
\begin{aligned}
\operatorname{Ker}\beta &= \{x\in K_A^\times \mid g(x)\in((\boldsymbol{Z}[G]/\boldsymbol{Z})\otimes U_K)^G\cdot((\boldsymbol{Z}[G]/\boldsymbol{Z})\otimes K^\times)^G\}/U_K\cdot K^\times \\
&= \{x\in K_A^\times \mid g(x)\in((\boldsymbol{Z}[G]/\boldsymbol{Z})\otimes U_K)^G\}\cdot K^\times/U_K\cdot K^\times.
\end{aligned}
$$

From the diagram (6), we have

$$
\begin{aligned}
g(x)\in((\boldsymbol{Z}[G]/\boldsymbol{Z})\otimes U_K)^G &\Longleftrightarrow \bar{g}(xU_K)=0 &&\text{in } ((\boldsymbol{Z}[G]/\boldsymbol{Z})\otimes I_K)^G \\
&\Longleftrightarrow x^{\sigma-1}\in U_K &&\text{for every } \sigma\in G.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\operatorname{Ker}\beta &\cong \{x\in K_A^\times \mid x^{\sigma-1}\in U_K \text{ for every } \sigma\in G\}\cdot K^\times/U_K/U_K\cdot K^\times/U_K \\
&\cong I_K^G\cdot P_K/P_K \cong I_K^G/P_K^G,
\end{aligned}
$$

where P_K is the principal ideal group of K. $I_K^G\cdot P_K/P_K$ is isomorphic to the group of ideal classes represented by ambiguous ideals in K/k. Then it is easy to show the following equation

$$[\operatorname{Ker}\beta]=[I_K^G : P_K^G]=\frac{[H^1(G, U_K)]h_k}{[H^1(G, O_K^\times)]}.$$

Theorem 2. *The following sequence is exact*

$$0\longrightarrow I_K^G/P_K^G\longrightarrow C(T)\xrightarrow{\beta} C(T'')\longrightarrow 0.$$

Corollary. *Let $a_{K/k}^0$ denote the order of the group $I_K^G\cdot P_K/P_K$. Then we have the equation $h_K=h'_{K/k}\cdot a_{K/k}^0$, where $h'_{K/k}$ is the class number of the torus $R_{K/k}(G_m)/G_m$.*

From the corollary, we have the equality

$$E'(K/k)=\frac{h_K}{h'_{K/k}\cdot h_k}=\frac{a_{K/k}^0}{h_k}=\frac{[H^1(G, U_K)]}{[H^1(G, O_K^\times)]}.$$

Remark. Prof. T. Ono has kindly informed the author of similar results by Mr. R. Sasaki obtained by another method. The author has also received a direct communication from Prof. V. E. Voskresenskii that he had got some results on the class number of algebraic tori, related implicitly to the author's results.

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