57. A Note on Modules

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Introduction. Let R be a fixed (not necessarily commutative) ring. Throughout this note, we are concerned with left R-modules $M, A, H, \cdots$. Like in Goldie [1], we shall use the following terminology. A non-zero submodule K of M is called *essential* in M (or M is an *essential extension* of K) if $K \cap A=0$ for any other submodule A of M, implies $A=0$. M has *finite Goldie dimension* (abbr. FGD) if M does not contain a direct sum of infinite number of non-zero submodules. Equivalently, M has finite Goldie dimension if for any strictly increasing sequence $H_0, H_1, \cdots$ of submodules of M, there is an integer i such that for every $k \geqslant i$, H_k is essential submodule in H_{k+1}. M is *uniform*, if every non-zero submodule of M is essential in M. Then it is proved (Goldie [1]) that in any module M with FGD, there exist non-zero uniform submodules $U_1, U_2, \cdots, U_n$ whose sum is direct and essential in M. The number n is independent of the uniform submodules. This number n is called the *Goldie dimension* of M and denoted by $\dim M$. It is easily proved that if M has FGD then every submodule of M has also FGD and $\dim K \leq \dim M$ (K being a submodule of M).

Furthermore, if K, A are submodules of M, and K is a maximal submodule of M such that $K \cap A=0$, then we say that K is a *complement* of A (or a complement in M). It is easily proved that if K is a complement in M, if and only if there exists a submodule A in M such that $A \cap K=0$ and $K' \cap A \neq 0$ for any submodule K' of M containing K. In this case we have $K+A$ is essential in M.

We are now introducing a notion "*E-irreducible submodule of M*". A submodule H of M is said to be E-irreducible if $H=K \cap J$, K and J are submodules of M, and H is essential in K, imply $H=K$ or $H=J$. Every complement submodule is an E-irreducible submodule, but the converse is not true.

Example 1. Consider Z, the ring of integers and Z_{12}, the ring of integers modulo 12. Write $R=Z$ and $M=Z_{12}$. Now the principal submodule K of M generated by 2, is E-irreducible submodule, but not a complement submodule.

Example 2. Consider $R=Z$ and $M=Z_8 \times Z_3$. Now the submodule $K=(4)\times(0)$ of M is not E-irreducible (since $K=(Z_8\times(0)) \cap ((4)\times Z_3)$ and K is essential in $Z_8\times(0)$).

The purpose of this note is to prove the following result.

Main theorem. *If K is a submodule of an R-module M and $f: M \to M/K$ is the canonical epimorphism, then the conditions given below are equivalent.*

(i) *$K=M$ or K is not essential, but E-irreducible.*

(ii) *K has no proper essential extensions.*

(iii) *K is a complement.*

(iv) *For any submodule K' of M containing K, K' is a complement in M if and only if $f(K')$ is a complement in M/K.*

(v) *$f(S)$ is essential in M/K for any essential submodule S of M.*

Moreover, if M has FGD then each of the above conditions (i)–(v) *are equivalent to*

(vi) *M/K has FGD and* $\dim(M/K)=\dim M-\dim K$.

2. Some results. We now list the results used in this paper.

Proposition 1. (i) *If K, K' are two submodules of M and K' is essential extension of K (that is, K is essential in K'), then* $\dim K=\dim K'$. (ii) *If A, B are two submodules such that the sum $A+B$ is direct, then* $\dim(A+B)=\dim A+\dim B$. (iii) *If M, N are two R-modules such that M is isomorphic to N, then* $\dim M=\dim N$. (iv) *A complement submodule has no proper essential extensions.* (v) *If A, B are two submodules such that $A \cap B=(0)$, then there exists a submodule C which is a complement of B containing A.* (vi) *Suppose K is a submodule of M. If K is not a complement, then there exists a complement submodule in M, which is a proper essential extension of K.*

Proposition 2 (Proposition 2, p. 61 [2]). *A module M is completely reducible if and only if M contains no proper essential submodules.*

3. Theorems. We devide our main theorem into three different theorems. In what follows, M will always mean a module.

Theorem 1. *Let K be a submodule of M and $f: M \to (M/K)$ be the canonical epimorphism. Then the following conditions are equivalent.*

(i) *K is a complement.*

(ii) *For any submodule K' of M containing K, K' is a complement in M if and only if $f(K')$ is a complement in M/K.*

(iii) *For any essential submodule S of M, $f(S)$ is essential in M/K.*

Proof. (i)$\Rightarrow$(ii) follows from the proof of Theorem 1.12 [1].

(ii)$\Rightarrow$(i): Since $f(K)=0$ is a complement in M/K, it is evident that K is a complement.

(i)$\Rightarrow$(iii): One can easily show this using the fact "K has no proper extensions".

(iii)$\Rightarrow$(i): Let X be a complement of K and K^* be a complement of X containing K. Now $X+K$ is essential in M and so $f(X)=f(X+K)$ is essential in M/K. Since $f(X) \cap f(K^*)=0$, we have $f(K^*)=0$ which shows $K=K^*$. This completes the proof of the theorem.

Theorem 2. *Let M be an R-module with finite Goldie dimension and K be a submodule of M such that* $\dim M=\dim K+\dim(M/K)$. *Then K*

has no proper essential extensions in M.

Proof. If $K=M$, there is nothing to prove. Suppose $K\neq M$ and K has a proper essential extension K'. So $\dim K=\dim K'$ and $\dim(K'/K)\geqslant 1$. Let C be a complement of K'. Then $C+K'$ is direct and essential in M. So we have $\dim M=\dim(C+K')=\dim C+\dim K'$. Since $((C+K)/K)$ is isomorphic with C, $\dim((C+K)/K)=\dim C$. Since the sum $(K'/K)+((C+K)/K)$ is direct, we have the following.

$$\begin{aligned}\dim(M/K)&\geqslant \dim(K'/K)+\dim((C+K)/K)\\&\geqslant 1+\dim C\\&=1+\dim M-\dim K'\\&=1+\dim M-\dim K\\&=1+\dim(M/K).\end{aligned}$$

This is a contradiction and hence K has no proper essential extensions.

Theorem 3. *Let K be a submodule of M. Then the following are equivalent.*

(i) *$K=M$ or K is not an essential submodule but it is an E-irreducible submodule.*

(ii) *K has no proper essential extensions.*

(iii) *K is a complement.*

Proof. (i)$\Rightarrow$(ii): If $K=M$, then there is nothing to prove. Suppose K is not essential, but E-irreducible. In a contrary way suppose K has a proper essential extension K'. We now show K' is essential in M. Let I be a submodule such that $K'\cap I=0$. By modular law $K'\cap(K+I)=K+(I\cap K')=K$. Since K is E-irreducible, $K\neq K'$ and K is essential in K', we have $K=K+I$, which implies $I\subseteq K\subseteq K'$. So $I=0$ and hence K' is essential in M. Since K is essential in K', we have K is also essential in M, a contradiction.

(ii)$\Rightarrow$(iii): Suppose K has no proper essential extensions. Let Z be a complement of K, and K' be a complement of Z containing K. Now K is essential in K' and by (ii), we have $K=K'$.

(iii)$\Rightarrow$(ii)$\Rightarrow$(i): Follows from the definitions.

Proof of the main theorem.

(i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii): Theorem 3.
(iii)$\Leftrightarrow$(iv)$\Leftrightarrow$(v): Theorem 1.
(iii)$\Rightarrow$(vi): Theorem 1.12 of [1].
(vi)$\Rightarrow$(ii): Theorem 2.

4. Applications. Combining our Main theorem and Proposition 2, we have the following equivalent conditions for a module M to be "Completely reducible".

Proposition 3. *If M is an R-module, then the following conditions are equivalent.*

(i) *M is a completely reducible module.*

(ii) *Every submodule of M is a complement submodule.*

(iii) *Every proper submodule of M is not an essential submodule but* *it is an E-irreducible submodule.*

(iv) *Every proper submodule of M has no proper essential extensions.*

(v) *For any submodule K of M with the canonical epimorphism $f: M \to M/K$, we have that: K' is a complement submodule in M if and only if $f(K')$ is a complement submodule in M/K.*

(vi) *For any submodule K of M with the canonical epimorphism $f: M \to M/K$, we have that: S is an essential submodule in M implies $f(S)$ is an essential submodule in M/K.*

Moreover, if G has FGD, then the above conditions are equivalent to each of the following.

(vii) *M has the descending chain condition on its submodules.*

(viii) *For any submodule K of M, M/K has FGD and*

$$\dim (M/K) = \dim M - \dim K.$$

Goldie proved: If M is an R-module with FGD then for any complement submodule K of M, the module M/K has FGD and $\dim M = \dim K + \dim (M/K)$. The converse of this result is a part of our main result.

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References

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[2] J. Lambek: Lectures on Rings and Modules. Blaisdell Publishing Company (1966).