# 56. *On Uniform Distribution of Sequences*

By P. KISS*),†) and R. F. TICHY**)

(Communicated by Shokichi IYANAGA, M. J. A., June 9, 1987)

Let $z$: $z_0=0<z_1<z_2\cdots$ be a subdivision of the interval $[0,\infty)$ with $z_n\to\infty$ as $n\to\infty$. For an increasing sequence $(x_n)_{n=1}^{\infty}$ of non-negative real numbers, define the sequence $(i_n)$ of positive integers by

$$z_{i_n-1}\leqq x_n<z_{i_n}.$$

Then $(x_n)$ is said to be uniformly distributed modulo the subdivision $z$ if the sequence

$$\{x_n\}_z=\frac{x_n-z_{i_n-1}}{z_{i_n}-z_{i_n-1}} \tag{1}$$

is uniformly distributed mod 1, i.e., if

$$\lim_{n\to\infty}(1/N)A(x,N,\{x_n\}_z)=x \qquad (0\leqq x\leqq 1), \tag{2}$$

where $A(x,N,\{x_n\}_z)$ denotes the number of indices $n$, $1\leqq n\leqq N$ such that $\{x_n\}_z$ is less than $x$.

The following distribution properties of the sequence $(x_n)=(n\theta)$ ($\theta$ an arbitrary positive real number) are well-known:

(i) If $z_n-z_{n-1}\to\infty$ and $z_n/z_{n-1}\to 1$ as $n\to\infty$, then $(x_n)$ is uniformly distributed mod $z$ (W. J. Le Veque [6]).

(ii) If $z_n-z_{n-1}$ is decreasing, then $(x_n)$ is uniformly distributed mod $z$ for almost all $\theta$; this result also holds in the case $(x_n)=(n^{\gamma}\theta)$ for any fixed $\gamma>0$ (H. Davenport and W. J. Le Veque [3]).

(iii) If $z_n/z_{n-1}\to 1$ as $n\to\infty$ and if the number of terms $z_n$ with $z_n\leqq N$ is less than $c\cdot N^{2-\delta}$ $(c,\delta>0)$, then $(x_n)$ is uniformly distributed mod $z$ for almost all $\theta$ (H. Davenport and P. Erdös [2]).

In the following we prove a generalization of some of these results by an elementary method (cf. [7]). For this purpose we define a sequence $(x_n)$ to be *almost uniformly distributed* mod $z$ if there is an infinite sequence $N_1<N_2<\cdots$ of positive integers such that

$$\lim_{n\to\infty}(1/N_i)A(x,N_i,\{x_n\}_z)=x \qquad (0\leqq x\leqq 1); \tag{3}$$

see Definitions 1.2 and 7.2 in the monograph of L. Kuipers and H. Niederreiter [5]. For further results on uniform distribution modulo a subdivision see Burkhard [1], P. Kiss [4].

**Theorem.** *Let $\theta$ be a positive real number and let $z=(z_n)$ be an increasing sequence of real numbers with conditions $z_0=0$ and $z_n/n\to\infty$ as $n\to\infty$. Then the sequence $(x_n)=(\theta n)$ $(n=1,2,\cdots)$ is almost uniformly distributed*

†) Research partially supported by Hungarian National Foundation for Scientific Research Grant No. 273.

*) Teachers' Training College, Eger, Hungary.

**) Department of Technical Mathematics, Technical University of Vienna, Vienna, Austria.

*modulo z. It is uniformly distributed* mod $z$ *if and only if* $\lim_{n\to\infty}(z_n/z_{n-1})=1$.

*Proof.* Let $x$ be a real number with $0<x<1$ and let

$$S_N=\sum_{n=1}^{M}\sum_{\substack{z_{n-1}<x_k\leqq z_n,\\ k<N}}\chi_{[0,x)}\left(\frac{x_k-z_{n-1}}{z_n-z_{n-1}}\right),$$

where $\chi_{[0,x)}$ is the characteristic function of the interval $[0,x)$ and $M=M(N)$ is an integer defined by

$$z_{M-1}<x_N\leqq z_M.$$

The definition of $M$ implies that there is a real number $\lambda$ $(0<\lambda\leqq 1)$ such that

$$N=(1/\theta)(z_{M-1}+\lambda(z_M-z_{M-1})). \tag{4}$$

Using the notation $\Delta z_n=z_n-z_{n-1}$ we derive from

$$0\leqq\frac{x_k-z_{n-1}}{\Delta z_n}<x$$

that

$$(1/\theta)z_{n-1}\leqq k<(1/\theta)(z_{n-1}+x\Delta z_n).$$

Hence we have

$$\sum_{z_{n-1}<x_k\leqq z_n}\chi_{[0,x)}\left(\frac{x_k-z_{n-1}}{\Delta z_n}\right)=\frac{x\Delta z_n}{\theta}+O(1) \tag{5}$$

for every $n$ with $n<M$.

Let first $\lambda\geqq x$. In this case, (5) holds also for $n=M$, and so

$$S_N=\sum_{n=1}^{M}\left(\frac{x\Delta z_n}{\theta}+O(1)\right)=\frac{xz_M}{\theta}+O(M).$$

Thus by (4) we obtain

$$\frac{S_N}{N}=\frac{xz_M+O(M)}{z_{M-1}+\lambda(z_M-z_{M-1})}=x\left(\frac{z_{M-1}}{z_M}(1-\lambda)+\lambda\right)^{-1}+O\left(\frac{M}{z_M}\right). \tag{6}$$

Now let $0<\lambda<x$. In this case we have

$$\sum_{\substack{z_{M-1}<x_k<z_M,\\ k<N}}\chi_{[0,x)}\left(\frac{x_k-z_{M-1}}{\Delta z_M}\right)=N-\frac{z_{M-1}}{\theta}+O(1)=\frac{1}{\theta}\lambda(z_M-z_{M-1})+O(1),$$

and so by (5)

$$S_N=\sum_{n=1}^{M-1}\frac{x\Delta z_n}{\theta}+\frac{1}{\theta}\lambda\Delta z_M+O(M)=\frac{1}{\theta}(xz_{M-1}+\lambda(z_M-z_{M-1}))+O(M).$$

Similarly as above we derive in this case

$$\frac{S_N}{N}=\frac{x+\lambda((z_M/z_{M-1})-1)}{1+\lambda((z_M/z_{M-1})-1)}+O\left(\frac{M}{z_M}\right). \tag{7}$$

By (6) and (7), since $M/z_M\to 0$ as $M\to\infty$,

$$\lim_{N\to\infty}S_N/N=x$$

does not depend on $\lambda$ if and only if $\lim_{M\to\infty}z_M/z_{M-1}$ exists and equals to 1. Thus the second assertion of the theorem is proved. Let $N_1, N_2, \cdots$ be the sequence of natural numbers defined by $N_i=[z_i/\theta]$, where $[\cdot]$ denotes the integer part function. For these integers, similarly as above we obtain

$$\frac{S_{N_i}}{N_i}=\frac{xz_{M(N_i)}+O(M(N_i))}{z_{M(N_i)}+O(1)}=x+O\left(\frac{M(N_i)}{z_{M(N_i)}}\right).$$

Hence

$$\lim_{i\to\infty} S_{N_i}/N_i = x$$

i.e. is $(x_n)$ is almost uniformly distributed mod $z$. This completes the proof of the theorem.

## References


[1] R. E. Burkhard: Zur Gleichmössigen Gleichrerteilung von Polynomwerten modulo $\Delta$. Sitzungsber. Österr. Akad. Wiss., math.-naturwiss. Kl., **177**, 395–406 (1969).

[2] H. Davenport and P. Erdös: A theorem on uniform distribution. Magyar Tud. Akad. Mat. Kutató Int. Közl., **8**, 3–11 (1963).

[3] H. Davenport and W. J. Le Veque: Uniform distribution relative to a fixed sequence. Michigan Math. J., **10**, 315–319 (1963).

[4] P. Kiss: Note on distribution of the sequence $n\theta$ modulo a linear recurrence. Discuss. Math., **7**, 135–139 (1985).

[5] L. Kuipers and H. Niederreiter: Uniform Distribution of Sequences. John Wiley, New York-London-Sydney-Toronto (1974).

[6] W. J. Le Veque: On uniform distribution modulo a subdivision. Pacific J. of Math., **3**, 757–771 (1953).

[7] Y. Ohkubo: Discrepancy with respect to weighted means of some sequences. Proc. Japan Acad., **62**A, 201–204 (1986).