52. On the Existence of Solutions for the Boundary Value Problem of Quasilinear Differential Equations on an Infinite Interval

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(Communicated by Kôsaku YOSIDA, M. J. A., June 9, 1987)

1. Introduction. In this paper we deal with the problem of the existence of solutions for the quasilinear differential system with a boundary condition

(1) $$x'=A(t,x)x+F(t,x)$$

(2) $$\mathcal{N}(x)=0.$$

Let A be a real $n\times n$ matrix continuous on $\boldsymbol{R}^+\times\boldsymbol{R}^n$, where $\boldsymbol{R}^+=[0,+\infty)$, and let F be an $\boldsymbol{R}^n$-valued function continuous on $\boldsymbol{R}^+\times\boldsymbol{R}^n$. We assume that $\mathcal{N}$ is a continuous operator from $C_r^{\lim}$ into $\boldsymbol{R}^n$ (not necessarily linear), where $C_r^{\lim}=\{x\in C(\boldsymbol{R}^+)\,;\,\lim_{t\to+\infty}x(t)$ exists and $\|x(t)\|\leqq r\}$. We consider the associated linear problem

(3) $$x'=B(t)x$$

(4) $$\mathcal{L}(x)=0.$$

Let B be a real $n\times n$ matrix continuous on $\boldsymbol{R}^+$ and let $\mathcal{L}$ be a bounded linear operator from $C^{\lim}$ into $\boldsymbol{R}^n$, where $C^{\lim}=\{x\in C(\boldsymbol{R}^+)\,;\,\lim_{t\to+\infty}x(t)$ exists and is finite$\}$. For example, $\mathcal{L}(x)=Px(0)-Q\lim_{t\to+\infty}x(t)$, where P, Q are known constant $n\times n$ matrices.

Hypothesis H$_1$. $\int_0^{+\infty}\|B(s)\|ds<+\infty$.

Hypothesis H$_2$. There exist no solutions for ((3), (4)) except for the zero solution.

In [1], Kartsatos required some qualitative conditions for A in (1) and proved the existence of solutions for ((1), (2)) under the conditions that Hypotheses H$_1$ and H$_2$ hold and that A, $\mathcal{N}$ are sufficiently close to B, $\mathcal{L}$ in some sense, respectively. However, the conditions for A are necessary ones if A is sufficiently close to B. We apply a different approach used in [2] and obtain an extention of [1].

2. Preliminaries. The symbol $\|\cdot\|$ will denote a norm in $\boldsymbol{R}^n$ and the corresponding norm for $n\times n$ matrices. Let $C(\boldsymbol{R}^+)$ be the space of $\boldsymbol{R}^n$-valued functions continuous and bounded on $\boldsymbol{R}^+$ with the supremum norm $\|\cdot\|_\infty$. Let $M(\boldsymbol{R}^+)$ be the space of real $n\times n$ matrices continuous and bounded on $\boldsymbol{R}^+$ with the supremum norm $\|A\|_\infty=\sup\{\|A(t)\|\,;\,t\in\boldsymbol{R}^+\}$. We put $\|\mathcal{L}\|=\sup\{\|\mathcal{L}(x)\|\,;\,\|x\|_\infty=1\}$ and $S_r=\{x\in\boldsymbol{R}^n\,;\,\|x\|\leqq r\}$.

We denote X_B by the fundamental matrix of solutions for (3) such that

$X_B(0)=I$, where I is the identity matrix. From the equalities

$$X_B(t)=I+\int_0^t B(s)X_B(s)ds \quad\text{and}\quad X_B^{-1}(t)=I-\int_0^t X_B^{-1}(s)B(s)ds,$$

it follows that

$$\|X_B(t)\|\leqq 1+\int_0^t \|B(s)\|\,\|X_B(s)\|ds \quad\text{and}\quad \|X_B^{-1}(t)\|\leqq 1+\int_0^t \|X_B^{-1}(s)\|\,\|B(s)\|ds.$$

Using Gronwall's lemma we obtain the following lemma.

Lemma 1. *Suppose that Hypothesis* H_1 *holds. Then*

$$\|X_B(t)\|\leqq K,\quad \|X_B^{-1}(t)\|\leqq K \quad \text{for } t\in \boldsymbol{R}^+, \tag{5}$$

where $K=\exp\left(\int_0^{+\infty}\|B(s)\|ds\right)$, *and there exist* $\lim_{t\to+\infty}X_B(t)$ *and* $\lim_{t\to+\infty}X_B^{-1}(t)$.

We denote U_B by the constant matrix such that $\mathcal{L}(X_B(\cdot)x_0)=U_Bx_0$ for any $x_0\in\boldsymbol{R}^n$. It is easy to prove the following lemma.

Lemma 2. *The following statements* (i)–(iii) *are equivalent.*

(i) *Hypothesis* H_2 *holds.*

(ii) *For each continuous* $\boldsymbol{R}^n$*-valued function* f *such that* $\int_0^{+\infty}\|f(s)\|ds<+\infty$ *and each* $c\in\boldsymbol{R}^n$, *there exists one and only one solution* $x\in C^{\lim}$ *for the problem*

$$x'=B(t)x+f(t),\qquad \mathcal{L}(x)=c.$$

(iii) $\det U_B\neq 0$.

The following lemma is an elementary result in linear algebra.

Lemma 3. *Suppose that* $\det U_B\neq 0$ *holds. Then there exists a positive number* $\rho<1$ *such that*

$$\|U_B^{-1}\|\leqq 1/\rho. \tag{6}$$

A set S in $C^{\lim}$ is said to be equiconvergent if for any $\varepsilon>0$ there exists a $T(\varepsilon)>0$ such that $\|f(t)-\lim_{\tau\to+\infty}f(\tau)\|<\varepsilon$ for all $f\in S$ and all $t\geqq T(\varepsilon)$. From Ascoli-Arzelà's theorem we have the following.

Lemma 4. *If a set* S *in* $C^{\lim}$ *is uniformly bounded, equicontinuous and equiconvergent, then* S *is relatively compact in* $C^{\lim}$.

3. Theorems. *We assume that Hypotheses* H_1 *and* H_2 *hold. From Lemmas* 1 *and* 2 *there exists a number* ρ *in* (6). *We assume that the following conditions* (7)–(9) *hold.*

$$\|A(t,x)-B(t)\|\leqq m_1(t) \qquad \text{for } (t,x)\in\boldsymbol{R}^+\times S_r. \tag{7}$$

$$\|F(t,x)\|\leqq m_2(t) \qquad \text{for } (t,x)\in\boldsymbol{R}^+\times S_r. \tag{8}$$

$$\|\mathcal{L}(x)-\mathcal{N}(x)\|\leqq ar \qquad \text{for } x\in C_r^{\lim}. \tag{9}$$

Here non-negative numbers δ, a, R *and measurable functions* m_1, m_2 *satisfy the following conditions* (10)–(13).

$$K^3\delta\exp(K^2\delta)\leqq\rho/\{\|\mathcal{L}\|\,\|U_B^{-1}\|\}. \tag{10}$$

$$aK\exp(\delta)<\rho(1-\rho). \tag{11}$$

$$R\leqq\frac{\rho(1-\rho)-aK\exp(\delta)}{\{K^2\exp(2\delta)\|\mathcal{L}\|+\rho(1-\rho)\}K\exp(\delta)}. \tag{12}$$

$$\int_0^{+\infty}m_1(s)ds\leqq\delta,\qquad \int_0^{+\infty}m_2(s)ds\leqq rR. \tag{13}$$

Now we consider the linear problem

$$x'=A(t,\ y(t))x+F(t,\ y(t)) \tag{14}$$

$$\mathcal{L}(x)=\mathcal{L}(y)-\mathcal{N}(y) \tag{15}$$

for $y\in C_r^{\mathrm{lim}}$. Let X_y be the fundamental matrix of solutions for (14) such that $X_y(0)=I$. From (7), (13) and Hypothesis H_1 it follows that $\int_0^{+\infty}\|A(s,\ y(s))\|ds<+\infty$ for $y\in C_r^{\mathrm{lim}}$. By applying Lemma 1 we have the constant matrix U_y such that $\mathcal{L}(X_y(\cdot)x_0)=U_yx_0$ for any $y\in C_r^{\mathrm{lim}}$ and any $x_0\in \boldsymbol{R}^n$.

We prove the existence and uniqueness of the solution for ((14), (15)).

Theorem 1. *Suppose that Hypotheses* H_1 *and* H_2 *hold. If the conditions* (7)–(13) *are satisfied, then for* $y\in C_r^{\mathrm{lim}}$ *there exists the inverse of* U_y *such that*

$$\|U_y^{-1}\|\leqq 1/\{\rho(1-\rho)\} \tag{16}$$

and there exists one and only one solution $x_y\in C_r^{\mathrm{lim}}$ *for* ((14), (15)).

Proof of Theorem 1. From the variation of parameters formula we have $X_y(t)=X_B(t)+X_B(t)\int_0^t X_B^{-1}(s)\{A(s,\ y(s))-B(s)\}X_y(s)ds$. By (5), (7) and (13)

$$\begin{aligned}&\|X_y(t)-X_B(t)\|\\&\quad\leqq K^2\int_0^t\|A(s,\ y(s))-B(s)\|\ \|X_y(s)-X_B(s)\|ds+K^3\delta\qquad \text{for } t\in \boldsymbol{R}^+.\end{aligned}$$

By (7), (10), (13) and Gronwall's lemma, we have for $t\in\boldsymbol{R}^+$

$$\begin{aligned}\|X_y(t)-X_B(t)\|&\leqq K^3\,\delta\exp\left(K^2\int_0^t\|A(s,\ y(s))-B(s)\|ds\right)\\&\leqq \rho/\{\|\mathcal{L}\|\|U_B^{-1}\|\}.\end{aligned}$$

Then for $x_0\in\boldsymbol{R}^n$

$$\begin{aligned}\|(U_B-U_y)x_0\|&\leqq\|\mathcal{L}\|\|X_B-X_y\|_\infty\|x_0\|\\&\leqq\rho\|x_0\|/\|U_B^{-1}\|.\end{aligned} \tag{17}$$

By (6) it follows that $\|U_yx_0\|\geqq\rho(1-\rho)\|x_0\|$ for $x_0\in\boldsymbol{R}^n$. Hence U_y has the inverse and (16) holds.

By Lemma 2 the problem ((14), (15)) has one and only one solution x_y such that

$$\begin{aligned}x_y(t)=&U_y^{-1}[\mathcal{L}(y)-\mathcal{N}(y)-\mathcal{L}(p_y(\cdot))]\\&+\int_0^t A(s,\ y(s))x_y(s)ds+\int_0^t F(s,\ y(s))ds,\end{aligned} \tag{18}$$

where $p_y(t)=X_y(t)\int_0^t X_y^{-1}(s)F(s,\ y(s))ds$ for $t\in\boldsymbol{R}^+$. By the same way in (5), $\|X_y(t)\|\leqq K\exp(\delta)$ and $\|X_y^{-1}(t)\|\leqq K\exp(\delta)$ for $t\in\boldsymbol{R}^+$. This yields $\|p_y\|_\infty\leqq rRK^2\exp(2\delta)$. From (18) we obtain

$$\|x_y(t)\|\leqq\frac{ar+\|\mathcal{L}\|rRK^2\exp(2\delta)}{\rho(1-\rho)}+rR+\int_0^t\|A(s,\ y(s))\|\|x_y(s)\|ds\qquad\text{for } t\in\boldsymbol{R}^+,$$

so that, by Gronwall's lemma,

$$\|x_y(t)\|\leqq\left(\frac{ar+\|\mathcal{L}\|rRK^2\exp(2\delta)}{\rho(1-\rho)}+rR\right)\exp\left(\int_0^t\|A(s,\ y(s))\|ds\right)\qquad\text{for } t\in\boldsymbol{R}^+.$$

Thus $\|x_y(t)\|\leqq r$ for $t\in\boldsymbol{R}^+$. This completes the proof.

By applying the fixed point theorem we obtain the following theorem.

Theorem 2. *Suppose that Hypotheses* H_1 *and* H_2 *hold. If the conditions* (7)–(13) *are satisfied, then there exists at least one solution for* ((1), (2)).

Proof of Theorem 2. It is easy to show that the solution x_y for ((14), (15)) can be expressed by

(19) $$x_y(t)=X_y(t)U_y^{-1}[\mathcal{L}(y)-\mathcal{N}(y)-\mathcal{L}(p_y(\cdot))]+p_y(t) \qquad \text{for } t\in \boldsymbol{R}^+.$$

Define $\mathcal{V}: C_r^{\mathrm{lim}}\to C_r^{\mathrm{lim}}$ by $\mathcal{V}(y)=x_y$ for $y\in C_r^{\mathrm{lim}}$, then $\mathcal{V}$ maps the closed convex set C_r^{lim} into itself.

Let $y_n\to y_0\ (n\to\infty)$ in C_r^{lim}. We have for $t\in \boldsymbol{R}^+$

$$X_{y_n}(t)=X_{y_0}(t)+X_{y_0}(t)\int_0^t X_{y_0}^{-1}(s)\{A(s,\ y_n(s))-A(s,\ y_0(s))\}X_{y_n}(s)ds.$$

This yields $\|X_{y_n}-X_{y_0}\|_\infty \leqq K^3\exp(3\delta)\int_0^{+\infty}\|A(s,\ y_n(s)-A(s,\ y_0(s))\|\,ds$. By Lebesgue's convergence theorem, $X_{y_n}\to X_{y_0}\ (n\to\infty)$ in $M(\boldsymbol{R}^+)$. By the same way in (17), $\|(U_{y_n}-U_{y_0})x_0\|\leqq\|\mathcal{L}\|\|X_{y_n}-X_{y_0}\|_\infty\|x_0\|$ for $x_0\in \boldsymbol{R}^n$. This implies $\|U_{y_n}-U_{y_0}\|\to 0\ (n\to\infty)$. From (16) $\|U_{y_n}^{-1}-U_{y_0}^{-1}\|\leqq\|U_{y_n}-U_{y_0}\|/\{\rho^2(1-\rho)^2\}$. We have $\|U_{y_n}^{-1}-U_{y_0}^{-1}\|\to 0\ (n\to\infty)$. By the same argument about the convergence of $\{X_{y_n}\}$, $X_{y_n}^{-1}\to X_{y_0}^{-1}\ (n\to\infty)$ in $M(\boldsymbol{R}^+)$. Therefore it follows that, by Lebesgue's convergence theorem,

$$\int_0^t X_{y_n}^{-1}(s)F(s,\ y_n(s))ds\to\int_0^t X_{y_0}^{-1}(s)F(s,\ y_0(s))ds \qquad (n\to\infty)$$

uniformly with respect to $t\in \boldsymbol{R}^+$. Thus, by (19),

$$\mathcal{V}(y_n)\to\mathcal{V}(y_0) \quad (n\to\infty) \quad \text{in } C_r^{\mathrm{lim}},$$

i.e., $\mathcal{V}$ is continuous on C_r^{lim}.

It is clear that $\mathcal{V}(C_r^{\mathrm{lim}})$ is uniformly bounded. By (18)

$$\|(\mathcal{V}(y))(t_1)-(\mathcal{V}(y))(t_2)\| \leqq \left|\int_{t_1}^{t_2}\{m_1(s)+\|B(s)\|\}r\,ds\right|+\left|\int_{t_1}^{t_2}m_2(s)ds\right| \qquad \text{for } t_1,\ t_2\in \boldsymbol{R}^+.$$

It follows that $\mathcal{V}(C_r^{\mathrm{lim}})$ is equicontinuous. By the same argument about the equicontinuity, $\mathcal{V}(C_r^{\mathrm{lim}})$ is equiconvergent. Thus, by Lemma 4, $\mathcal{V}(C_r^{\mathrm{lim}})$ is a relatively compact set in C^{lim}.

According to Schauder's fixed point theorem, $\mathcal{V}$ has at least one fixed point in C_r^{lim}. Therefore there exists at least one solution for ((1), (2)), and this completes the proof.

References

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