50. *Limiting Behaviour of Linear Cellular Automata*

By Satoshi TAKAHASHI
Department of Biophysics, Kyoto University

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1. Introduction. Stephan J. Willson pointed out that cellular automata can generate fractals [1]. Specifically he showed that in any 2-state linear cellular automaton a sequence of space time pattern at time 2^n contracted by a factor $1/2^n$ converges to a limit when n tends to infinity. This result, however, can be generalized, and we will show in this paper that every linear cellular automaton has its "limit set". (Limit set is defined in section 2.)

2. Preliminaries. Throughout this paper, we assume that p is a prime and $t, d, M, m, k \in N$ where N is the set of natural numbers.

A d-dimensional M-state cellular automaton can be defined as follows. Consider d-dimensional lattice points on which copies of an automaton called "cells" are located. Let $i \in Z^d$ denote the site of a cell, and t discrete time step. Each cell takes a state value which belongs to $\{0, \cdots, M-1\}$. We denote by a_i^t the state of an i-cell at time t. The states of cells at time t are determined by the states of cells at time $t-1$ by,

$$a_i^t = f(a_{i-j^1}^{t-1}, a_{i-j^2}^{t-1}, \cdots, a_{i-j^m}^{t-1})$$

where $j^1, j^2, \cdots, j^m \in Z^d$ and $f: \{0, \cdots, M-1\}^m \to \{0, \cdots, M-1\}$. f is a "local transient function" and $\{j^1, \cdots, j^m\}$ is called a "neighbourhood index".

We treat only the case where f is "linear" i.e.,

$$f(x_1, x_2, \cdots, x_m) = \alpha_1 \cdot x_1 + \cdots + \alpha_m \cdot x_m \bmod M \quad (\alpha_1, \cdots, \alpha_m \in N).$$

The space time pattern of a cellular automaton sometimes reveals interesting properties. To study it in its limit, we define "limit set" of a cellular automaton as follows.

Definition 2.1. Let $T: N \to N$ and $T(1) < T(2) < T(3) \cdots$. We define $S(n)$, $\lim S \subseteq R^{d+1}$ for this $T(n)$ as follows.

$$S(n) = \left\{ \left(\frac{t}{T(n)}, \frac{i_1}{T(n)}, \frac{i_2}{T(n)}, \cdots, \frac{i_d}{T(n)} \right) \middle| a_i^t \neq 0,\ t \leqq T(n) \text{ and } i = (i_1, \cdots, i_d) \right\}.$$

We define limsup $S(n)$ and liminf $S(n)$ as in S. J. Willson ([1], p. 93).

If limsup $S(n)$ = liminf $S(n)$, then we define $\lim S$ the "limit set" of cellular automaton by,

$$\lim S = \operatorname{limsup} S(n) = \operatorname{liminf} S(n).$$

3. Existence of limit set. Unless otherwise stated, we assume that $a_i^0 = 1$ for $i = (0, \cdots, 0)$ and $a_i^0 = 0$ otherwise.

Some properties of linear cellular automata are derivable from those of multinomial coefficients.

Multinomial coefficients have, for example, the following properties (see [2] Theorem 3.10, [3] p. 380).

Lemma 3.1. *Suppose* $t=c_1+c_2+\cdots+c_m$, $p^{k+s}|t$, *and* $\exists u\in N$ $(1\leqq u\leqq m)$ $p^{s+1}\nmid c_u$. *Then we have,*

$$\frac{t!}{c_1!\,c_2!\cdots c_m!}\equiv 0 \bmod p^k.$$

Lemma 3.2. *Suppose* $t=c_1+c_2+\cdots+c_m$ *and* $p^{2k-2}|t$ *then,*

$$\frac{(p\cdot t)!}{(p\cdot c_1)!\cdot(p\cdot c_2)!\cdot\cdots\cdot(p\cdot c_m)!}\equiv\frac{t!}{c_1!\cdot c_2!\cdot\cdots\cdot c_m!} \bmod p^k.$$

From these properties of multinomial coefficients, we can show the following theorem.

Theorem 3.3. *Linear* p^k*-state cellular automata have the following properties.*

(1) *If* $p^{k+s}|t$ *and* $\exists u\in N$ $(1\leqq u\leqq d)$, $p^{s+1}\nmid i_u$ *where* $i=(i_1,\cdots,i_d)$, *then* $a_i^t=0$.

(2) *If* $p^{2k-2}|t$ *then* $a_{pi}^{pt}=a_i^t$.

Proof. By the linearity, we note first that

$$a_i^t\equiv\sum_{c_1\cdots c_m}\frac{t!}{c_1!\cdot\cdots\cdot c_m!}\alpha_1^{c_1}\cdot\alpha_2^{c_2}\cdot\cdots\cdot\alpha_m^{c_m} \bmod M \tag{2.1}$$

$$(c_1,\cdots,c_m\in N).$$

The sum is over all $(c_n,\cdots,c_m)$ such that

$$t=c_1+\cdots+c_m,\quad\text{and}\quad c_1\cdot j^1+\cdots+c_m\cdot j^m=i.$$

If $p^{k+s}|t$ and $\exists u\in N$ $(1\leqq u\leqq d)$, $p^{s+1}\nmid i_u$, then we deduce that $\exists c_v$ $(1\leqq v\leqq m)$ $p^{s+1}\nmid c_v$ for $(c_1,\cdots,c_m)$ of (2.1), and therefore $a_i^t=0$ from Lemma 3.1. So (1) is proved.

We write down also a_{pi}^{pt} as in (2.1).

$$a_{pi}^{pt}=\sum_{c_1\cdots c_m}\frac{(p\cdot t)!}{(p\cdot c_1)!\cdot\cdots\cdot(p\cdot c_m)!}\alpha_1^{pc_1}\cdot\cdots\cdot\alpha_m^{pc_m}$$
$$+\sum_{s_1\cdots s_m}\frac{(p\cdot t)!}{s_1!\cdot\cdots\cdot s_m!}\alpha_1^{s_1}\cdot\cdots\cdot\alpha_m^{s_m} \bmod M$$

where the first term is summed under the restriction

$$t=c_1+\cdots+c_m,\quad\text{and}\quad c_1\cdot j^1+\cdots+c_m\cdot j^m=i,$$

while the second term is summed over all $(s_1,\cdots,s_m)$ which satisfies

$$(p\cdot t)=s_1+\cdots+s_m,\ s_1\cdot j^1+\cdots+s_m\cdot j^m=(p\cdot i)\quad\text{and}\quad \exists s\ (1\leqq u\leqq m)p\nmid s_u.$$

From Lemma 3.1, the second term equals 0.

So, Euler's theorem ($\forall r\in N r^{p^k}\equiv r^{p^{k-1}} \bmod p^k$) and Lemma 3.2 give the relation (2) directly. This completes the proof.

For p^k-state linear cellular automata, we define a subset $X(n)$ of $S(n)$ as follows.

$$X(n)=\left\{\left(\frac{t}{T(n)},\frac{i_1}{T(n)},\frac{i_2}{T(n)},\cdots,\frac{i_d}{T(n)}\right)\Big|\, a_i^t\neq 0,\ t\leqq T(n),\ p^{2k-2}|t,\ i=(i_1,\cdots,i_d)\right\}.$$

Then $\lim X=\limsup X(n)=\liminf X(n)$ when $\limsup X(n)=\liminf X(n)$.

If we assume that $T(n)=p^n$, then we have $X(1)\subseteq X(2)\subseteq\cdots$ from Theorem 3.3. So there exists $\lim X$.

The following theorem can be verified using a similar method to that of Willson ([1], p. 94).

Theorem 3.4. *For any M-state linear cellular automaton where $M=p^k$ (p denotes a prime, $k\in N$), a limit set* $\lim S$ *exists for* $T(n)=p^n$. *Furthermore this* $\lim S$ *is equal to* $\lim X$.

This theorem can be generalized to the following.

Theorem 3.5. *Given any linear cellular automaton, there exists a $T(n)$ for which a limit set of this cellular automaton exists.*

4. Application. We can relax the restriction for an initial state configuration stated in the beginning of section 3.

Theorem 4.1. *Consider an M-state cellular automaton with an initial state configuration where only a finite number of cells have nonzero state values. Then the limit set of this cellular automaton is equal to the limit set of an M′-state cellular automaton which has the same transition function as before except that $M'=\underset{i}{L.C.M.}\,(M/(M, a_i^0))$, where (M, a_i^0) means G. C. D. of M and a_i^0.*

Concerning Hausdorff dimension, we have,

Theorem 4.2. *Consider a cellular automaton with $M=p^k$. Assume $p\nmid\alpha_1, \cdots, p\nmid\alpha_r, p|\alpha_{r+1}, \cdots, p|\alpha_m$, and $(1, j^1), \cdots, (1, j^r)\in Z^{d+1}$ are linearly independent, then the Hausdorff dimension of a limit set of this cellular automaton is* $\log_p(p\cdot(p+1)\cdot\cdots\cdot(p+r-1)/r!)$.

We also investigated the limiting behaviour of a particular state. We define $S(n\,;b)$, $\lim S(b)$ for $1\leqq b\leqq M-1$, as in Definition 2.1.

$$S(n\,;b)=\left\{\left(\frac{t}{T(n)}, \frac{i_1}{T(n)}, \frac{i_2}{T(n)}, \cdots, \frac{i_d}{T(n)}\right)\middle| a_i^t=b \bmod M,\ t\leqq T(n)\right\}$$

$$\lim S(b)=\limsup S(n\,;b)=\liminf S(n\,;b)$$

when

$$\limsup S(n\,;b)=\liminf S(n\,;b).$$

Then we have,

Theorem 4.3. *For a given M-state cellular automaton, assume that $M=p^k, p^r|b, p^{r+1}\nmid b$, and $\exists u, v\in N(1\leqq u, v\leqq m, u\neq v)\ p\nmid\alpha_u\ p\nmid\alpha_v$. Then* $\lim S(b)$ *is equal to the limit set of another M′-state cellular automaton which has the same transition function except that $M'=p^{k-r}$.*

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