49. A Result on the Scattering Theory for First Order Systems with Long-range Perturbations

By Koji KIKUCHI
Department of Mathematics, Osaka University

(Communicated by Kôsaku YOSIDA, M. J. A., June 9, 1987)

In this report we treat the following differential equation for C^m-valued function:

$$D_t u = \Lambda u,$$

where $D_t=(1/i)(\partial/\partial t)$ and

$$\Lambda = E(x)^{-1/2}\sum_{j=1}^{n} A_j D_j E(x)^{-1/2}, \tag{1}$$

A_j's are $m\times m$ constant hermitian matrices, and $E(x)$ is a continuous $m\times m$ hermitian matrix valued function with

$$0<c_1 I\leq E(x)\leq c_2 I$$

for some constants c_1 and c_2. Λ can be extended to a self-adjoint operator on $\mathcal{H}=L^2(\boldsymbol{R}^n)$. If we substitute $E(x)$ with I in (1), we have a differential operator of constant coefficients:

$$\Lambda^0=\sum_{j=1}^{n} A_j D_j.$$

Λ^0 can also be extended to a self-adjoint operator on $\mathcal{H}$, and Λ is regarded as a perturbed operator of Λ^0. The main result which we shall report here is the existence theorem of the wave operator between Λ^0 and Λ. We consider the case that the perturbation is long-range. More precisely we assume that

Assumption (E). 1) $E(x)\in C^\infty(\boldsymbol{R}^n)$.

2) $|\partial_x^\alpha(E(x)-I)|\leq(1+|x|)^{-\delta-|\alpha|}$ for $\delta>0$ and $|\alpha|\geq 0$.

The operator $W_\pm$ is called the wave operator if the limit

$$W_\pm u=\lim_{t\to\pm\infty} e^{it\Lambda}e^{-it\Lambda^0}u \qquad (u\in\mathcal{H}_{ac}(\Lambda^0)) \tag{2}$$

exists. In the case of the short-range ($\delta>1$) it is already known that, for wide class of Λ^0, $W_\pm$ exists and is complete (see for example [3]). But it does not exist generally when the perturbation is long-range ($0<\delta\leq 1$). Then we should consider the modified wave operator. The fundamental problems of the theory of long-range perturbation are the existence and completeness of the modified wave operator. However few works have been treated related to the spectral theory of systems with long-range perturbations. There are only the works related to the limiting absorption principle ([3], [4]). Then unlike the case of the short-range the existence theorem is the first step of this theory.

On Λ^0 we assume the following. We put

$$\Lambda^0(\xi)=\sum_{j=1}^{n} A_j\xi_j \qquad \text{(symbol of } \Lambda^0\text{)}.$$

Then

Assumption (F). 1) Λ^0 is strongly propagative, that is, for some d

$$\operatorname{rank} \Lambda^0(\xi)=m-d \qquad \text{when} \quad \xi\neq 0.$$

2) We put

$$\rho_0=\max_{\xi\in \boldsymbol{R}^n} \sharp\,\{\text{distinct positive eigenvalues of } \Lambda^0(\xi)\},$$

and

$$\rho_0=(m-d)/2.$$

Remark. The condition 2) is equivalent to that the multiplicities are all simple for almost all $\xi\in \boldsymbol{R}^n$.

Example (Maxwell equation). We consider the Maxwell equation in crystals:

$$\nabla\times \boldsymbol{H}-\varepsilon(\partial \boldsymbol{E}/\partial t)=0, \qquad \nabla\times \boldsymbol{E}+\mu_0(\partial \boldsymbol{H}/\partial t)=0,$$

where $\varepsilon=(\varepsilon_{ij})$ is a tensor dielectric constant and μ_0 is a scalar magnetic permeability. Let ε_1, ε_2 and ε_3 be eigenvalues of ε. We may assume that

$$\varepsilon_1\geq\varepsilon_2\geq\varepsilon_3>0.$$

There are three classes which are defined by the condition (i) $\varepsilon_1>\varepsilon_2>\varepsilon_3$, (ii) $\varepsilon_1>\varepsilon_2=\varepsilon_3$ or $\varepsilon_1=\varepsilon_2>\varepsilon_3$ and (iii) $\varepsilon_1=\varepsilon_2=\varepsilon_3$ (isotropic). The classes (i) and (ii) satisfy Assumption (F), and the class (iii) does not satisfy (refer to C. H. Wilcox [5]).

The eigenvalues of $\Lambda^0(\xi)$ can be enumerated as follows:

$$\lambda^0_{\rho_0}(\xi)\geq\cdots\geq\lambda^0_1(\xi)>\lambda^0_0(\xi)\equiv 0>\lambda^0_{-1}(\xi)\geq\cdots\geq\lambda^0_{-\rho_0}(\xi).$$

And $\hat{P}^0_k(\xi)$ denotes the projection onto the eigenspace associated with $\lambda^0_k(\xi)$. $\bar{Z}^{(1)}_S$ denotes a set given by

$$\bar{Z}^{(1)}_S=\{\xi\in \boldsymbol{R}^n\,;\ \lambda^0_j(\xi)=\lambda^0_k(\xi) \text{ for some } j\neq k\}.$$

We put $\Lambda(x,\xi)=E(x)^{-1/2}\Lambda^0(\xi)E(x)^{-1/2}$ and

$$\rho(x)=\max_{\xi\in \boldsymbol{R}^n} \sharp\,\{\text{distinct positive eigenvalues of } \Lambda(x,\xi)\}.$$

Then the eigenvalues of $\Lambda(x,\xi)$ are also enumerated as

$$\lambda_{\rho(x)}(x,\xi)\geq\cdots\geq\lambda_1(x,\xi)>\lambda^0_0(\xi)\equiv 0$$
$$>\lambda_{-1}(x,\xi)\geq\cdots\geq\lambda_{-\rho(x)}(x,\xi).$$

$\hat{P}_k(x,\xi)$ and $\bar{Z}^{(1)}_{S_x}$ can also be defined similarly.

Then we have

Proposition. *Under Assumptions* (E) *and* (F) *the following facts hold:*

i) *There exists an* $R>0$ *such that*

$$\rho(x)=\rho_0 \qquad \text{when} \quad |x|>R.$$

ii) *There exists a family of open neighborhoods* $\{V_r\}_{r>R}$ *of* $\bar{Z}^{(1)}_S$ *such that*

$$\bigcup_{r>R} V_r=\bar{Z}^{(1)}_S$$

and

$$\bar{Z}^{(1)}_{S_x}\subset V_r \qquad \text{for} \quad |x|=r.$$

iii) *Let* $K\subset \boldsymbol{R}^n\backslash \bar{Z}^{(1)}_S$ *be compact. Take* R *of* i) *so large that* $K\cap V_r=\emptyset$ *if* $r>R$. *Then*

$$|\partial^\alpha_\xi\partial^\beta_x(\lambda_k(x,\xi)-\lambda^0_k(\xi))|\leq C_\xi\langle x\rangle^{-\delta-|\beta|}$$

for $|\alpha|, |\beta|\geq 0$ *and* $(x, \xi)\in\{|x|>R\}\times K$.

iv) $$|\partial_\xi^\alpha\partial_x^\beta(\hat{P}_k(x,\xi)-\hat{P}_k^0(\xi))|\leq C_\xi\langle x\rangle^{-\delta-|\beta|}$$

for $|\alpha|, |\beta|\geq 0$ *and* $(x, \xi)\in\{|x|>R\}\times K$.

Here C_ξ *is uniform for* $(x,\xi)\in\{|x|>R\}\times K$.

As stated above the limit (2) does not exist when the perturbation is long-range. We seek the modified wave operator as the following form:

$$W_\pm^D u=\lim_{t\to\pm\infty} e^{it\Lambda}X_t u \qquad (u\in\mathcal{H}_{ac}(\Lambda^0)).$$

Here X_t is an operator on $\mathcal{H}$ given by

$$X_t u=\sum_{|k|=1}^{\rho_0}(2\pi)^{-n}\int_{\Omega_t} e^{ix\xi-iW_k(t,\xi)}\hat{P}_k(\nabla W_k(t,\xi),\xi)\hat{u}(\xi)d\xi,$$

where $W_k(t,\xi)$'s are functions constructed in the lemma given soon later and Ω_t's are open domains defined by

$$\Omega_t=\Omega_l \qquad \text{for } t\in[t_l, t_{l+1})$$

with $\{t_l\}$ and $\{\Omega_l\}$ given in the same lemma.

Then we state the lemma in the case of $t\to\infty$.

Lemma. *Under Assumptions* (E) *and* (F), *let* $\{t_l\}_{l=1}^\infty$ *be a given sequence with* $t_1<t_2\cdots$ *and* $\lim_{l\to\infty}t_l=\infty$. *Then there exists a sequence of conic open sets* $\{\Omega_l\}_{l=0}^\infty$ *with* $\Omega_0\subset\Omega_1\subset\cdots$ *and* $\bigcup_{l=0}^\infty\Omega_l=\mathbf{R}^n\backslash\bar{Z}_S^{(1)}$, *and there exists a solution of the Hamilton-Jacobi equations*

$$\partial W_k/\partial t=\lambda_k(\nabla W_k,\xi) \qquad (|k|=1,2,\cdots,\rho)$$

on $\bigcup_{l=1}^\infty[t_l,\infty)\times\Omega_l$ *which satisfies, for any* $\xi\in K\subset\mathbf{R}^n\backslash\bar{Z}_S^{(1)}$ *a compact set.*

1) *If t is sufficiently large,* $V_{|\nabla W_k|}\cap K=\emptyset$.
2) $|\partial_\xi^\alpha W_k(t,\xi)|\leq C_\alpha t$ *for* $|\alpha|\geq 1$.
3) $|\partial_\xi^\alpha(t^{-1}W_k(t,\xi)-\lambda_k^0(\xi))|+|\partial_\xi^\alpha(\lambda_k(\nabla W_k,\xi)-\lambda_k^0(\xi))|\leq C_\alpha t^{-\delta}$ *for* $|\alpha|\geq 0$.
4) *For* $r>0$, $W_k(t, r\xi)=rW_k(t,\xi)$.
5) $W_k(t,-\xi)=-W_{-k}(t,\xi)$.

The proof of this lemma is similar to that of Theorem 3.8 of [1].

The main result is stated as the following.

Theorem. *The modified wave operator*

$$W_\pm^D=\text{s-}\lim_{t\to\pm\infty}e^{it\Lambda}X_t$$

exists, and it is a partial isometric operator with the intertwining property

$$e^{is\Lambda}W_\pm^D u=W_\pm^D e^{is\Lambda^0}u \qquad \textit{for } s\in\mathbf{R} \textit{ and } u\in\mathcal{H}_{ac}(\Lambda^0).$$

Details and the proof of this theorem are given in [2].

References

[1] L. Hörmander: The existence of wave operators in scattering theory. Math. Z., **146**, 69–91 (1976).

[2] K. Kikuchi: Scattering theory for first order systems with long-range perturbations: The existence of the modified wave operators (preprint).

[3] H. Tamura: Spectral and scattering theory for symmetric systems of nonconstant deficit. J. Funct. Anal., **67**, 73–104 (1986).

[4] R. Weder: Spectral analysis of strongly propagative systems. J. Angewandte Math., **354**, 95–122 (1984).

[5] C. H. Wilcox: Electromagnetic signal propagation in crystals. Appl. Anal., **8**, 83–94 (1978).