48. On the Highest Degree of Absolute Polynomials of Alternating Links

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§1. **Statement of results.** Let L be a tame link in S^3. For a regular projection $\tilde{L}$ of L, $c(\tilde{L})$ denotes the number of crossings of $\tilde{L}$ and $c(L)$ denotes the minimum number of crossings among all regular projections of L. $\tilde{L}$ divides S^2 into finitely many domains, which can be colored by two colors (black and white) like a chess-board such that domains of the same color meet only at crossing points. Let $g(\tilde{L})$ and $g^*(\tilde{L})$ be the graphs of $\tilde{L}$ such that vertices of $g(\tilde{L})$ and $g^*(\tilde{L})$ correspond to the white and the black domains, respectively, and each edge of $g(\tilde{L})$ and $g^*(\tilde{L})$ corresponds to a crossing of $\tilde{L}$. A vertex v of a graph is called a *stump* if the valency of v is equal to one, and a *twig* if the valency of v is equal to two.

R. D. Brandt, W. B. R. Lickorish and K. C. Millett defined in [1] an unoriented link invariant $Q(L)$ called the *absolute polynomial*. We refer the reader to [1] for the details.

In this paper we prove the following

Theorem. *Let L be a tame alternating link and $\tilde{L}$ a regular alternating projection of L. Then the following conditions are equivalent:*
(1) *The graphs $g(\tilde{L})$ and $g^*(\tilde{L})$ are connected without stumps, loops and cut-vertices.*
(2) *The highest degree of the absolute polynomial $Q(L)$ of L is equal to $c(\tilde{L})-1$, and the coefficient of the term of the highest degree of $Q(L)$ is positive.*

K. Murasugi proved in [5] that if $\tilde{L}$ is a regular connected proper alternating projection of an alternating link L, then the reduced degree of Jones polynomial [2] of L is equal to $c(\tilde{L})$ and $c(\tilde{L})=c(L)$. If L is a prime alternating link, then L has a regular projection $\tilde{L}$ which satisfies the condition (1) of Theorem. Therefore we have

Corollary 1. *If L is a prime alternating link then the highest degree of $Q(L)$ is equal to $c(L)-1$.*

The following is a part of Theorem 1 of W. Menasco [4], for which we will give an alternative proof.

Corollary 2. *If L_1 and L_2 are alternating links and $L=L_1 \# L_2$ is also an alternating link, then, for any connected regular proper alternating projection $\tilde{L}$ of L, $g(\tilde{L})$ and $g^*(\tilde{L})$ have cut-vertices.*

In [3], M. E. Kidwell independently obtained the similar results to our theorem. His method of the proof is different from ours.

§ 2. **Proofs.** We work in the PL category and all projections of links we consider are assumed to be regular.

The absolute polynomial $Q(L)$ has the following properties:

Theorem A ([1, Properties 1 and 8]). *Let L, L_1 and L_2 be tame links in S^3 then the following formulae hold.*

(a) $Q(L_1 \# L_2) = Q(L_1)Q(L_2)$, *where $L_1 \# L_2$ denotes any connected sum of L_1 and L_2.*

(b) $Q(L_1 \circ L_2) = \mu Q(L_1)Q(L_2)$, *where $\mu = 2x^{-1} - 1$ and $L_1 \circ L_2$ denotes split union.*

(c) $h\text{-deg}\, Q(L) \leqq c(L) - 1$, *where $h\text{-deg}\, Q(L)$ denotes the highest degree of $Q(L)$.*

To prove Theorem, we prove the following lemmas.

Lemma 1. *We suppose that $\tilde{L}$ is an alternating projection of a link L such that $c(\tilde{L}) \geqq 3$ and that $g(\tilde{L})$ is a connected graph without stumps, loops and cut-vertices. Let $\tilde{L}_0$ and $\tilde{L}_\infty$ be the projections in* [1, Theorem]. *Then $\tilde{L}_0$ and $\tilde{L}_\infty$ are again alternating projections and either $g(\tilde{L}_0)$ or $g(\tilde{L}_\infty)$ is a connected graph without stumps, loops and cut-vertices.*

Proof. Fig. 1 shows that $\tilde{L}_0$ and $\tilde{L}_\infty$ are alternating projections.

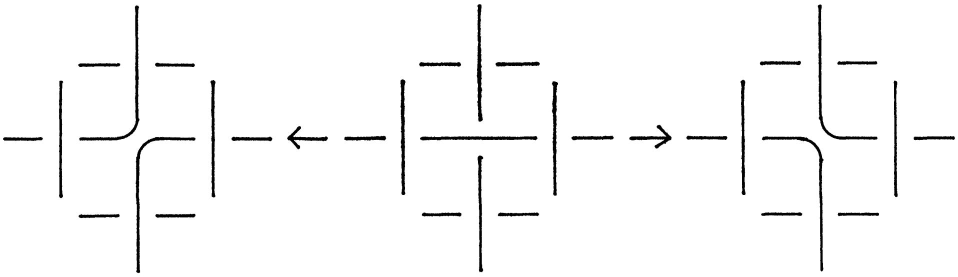

Fig. 1

Without loss of generality, we may assume that $\tilde{L}_0$ and $\tilde{L}_\infty$ are as shown in Fig. 2.

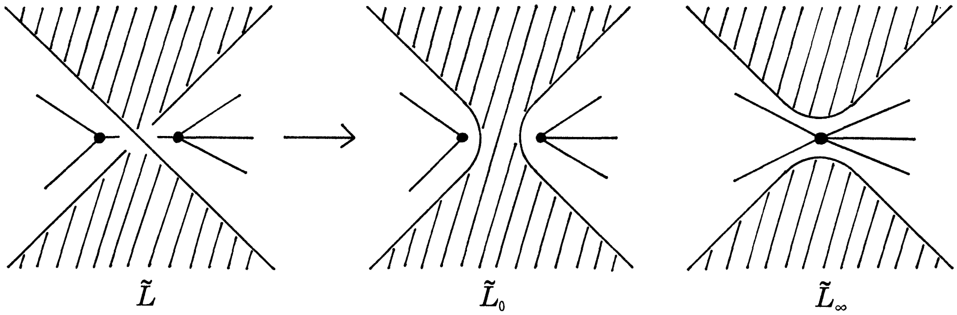

Fig. 2

By the condition of $g(\tilde{L})$, both $g(\tilde{L}_0)$ and $g(\tilde{L}_\infty)$ are connected, and $g(\tilde{L}_0)$ has no loops and $g(\tilde{L}_\infty)$ has no stumps. There are two cases to be considered.

Case 1. The graph $g(\tilde{L}_0)$ has a stump: In this case, $g(\tilde{L})$ has a twig. Since $c(\tilde{L}) \geqq 3$ and $g(\tilde{L})$ has no cut-vertices, $g(\tilde{L})$ is as in Fig. 3.

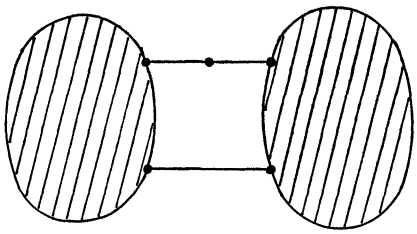

Fig. 3

Then $g(\tilde{L}_\infty)$ has none of stumps, loops and cut-vertices.

Case 2. $g(\tilde{L}_0)$ has a cut-vertex: In this case $g(\tilde{L}_\infty)$ has none of stumps, loops and cut-vertices. See Fig. 4. □

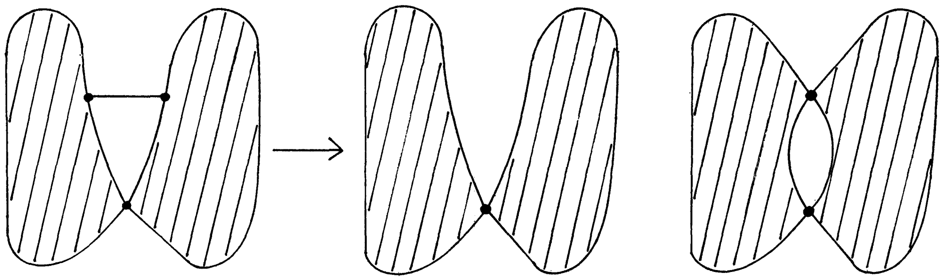

Fig. 4

Lemma 2. *If $\tilde{L}$ is a non-alternating projection of a link L, then*

$$h\text{-deg}\, Q(L) \leqq c(\tilde{L}) - 2.$$

Proof. If $\tilde{L}$ is disconnected, then by Theorem A the assertion holds. Therefore it is sufficient to consider the case $\tilde{L}$ is connected. Since $\tilde{L}$ is a non-alternating projection, we may assume that $\tilde{L}$ has two successive under crossing points p_1 and p_2.

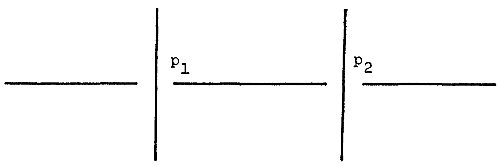

Fig. 5

We can obtain a sequence of link projections $\tilde{L} = \tilde{L}_0 \to \tilde{L}_1 \to \tilde{L}_2 \to \cdots \to \tilde{L}_m$ such that $\tilde{L}_{i+1}$ is obtained from $\tilde{L}_i$ by switching one of the crossings except p_1 and p_2 and that $\tilde{L}_m$ is an ascending projection of $\tilde{L}$. Since we do not switch the crossings p_1 and p_2 in all $\tilde{L}_i$'s, all $\tilde{L}_i$'s are non-alternating projections.

Hence we can obtain a resolution R of $\tilde{L}$ such that all projections of R are non-alternating projections. Since a non-alternating projection whose number of crossings is two is a trivial link, we can obtain trivial links by smoothing at most $c(\tilde{L})-2$ crossings. Hence h-deg $Q(L) \leqq c(\tilde{L})-2$. $\square$

Proof of Theorem. First, we prove necessity. If $g(\tilde{L})$ has a stump or a loop, then there exists the projection $\tilde{L}'$ of L such that $c(\tilde{L}')=c(\tilde{L})-1$. Therefore h-deg $Q(L) \leqq c(\tilde{L}')-1=c(\tilde{L})-2$. If $g(\tilde{L})$ is disconnected then $\tilde{L}=\tilde{L}_1 \cup \tilde{L}_2 \cup \cdots \cup \tilde{L}_n$, where $\tilde{L}_i$ is a connected component and $n \geqq 2$. By Theorem A we have h-deg $Q(L)=\sum_{i=1}^n h$-deg $Q(L_i) \leqq \sum_{i=1}^n (c(\tilde{L}_i)-1)=\sum_{i=1}^n c(\tilde{L}_i)-n=c(\tilde{L})-n<c(\tilde{L})-1$. If $g(\tilde{L})$ has a cut-vertex then $g(\tilde{L})$ is one point union of subgraphs Γ_1 and Γ_2. Let $\tilde{l}_i$ denote the projection corresponding to Γ_i $(i=1, 2)$. Since L is a split or connected sum of links l_1 and l_2, we have h-deg $Q(L)=\sum_{i=1}^2 h$-deg $Q(l_i) \leqq \sum_{i=1}^2 (c(\tilde{l}_i)-1)=\sum_{i=1}^2 c(\tilde{l}_i)-2 <c(\tilde{L})-1$.

Now we prove sufficiency of Theorem by induction on $c(\tilde{L})$. In case $c(\tilde{L})=2$, L is the Hopf link and $Q(L)=-2x^{-1}+1+2x$. In case $c(\tilde{L}) \geqq 3$, by Lemma 1, at least one of $\tilde{L}_0$ and $\tilde{L}_\infty$ has a connected graph without stumps, loops, and cut-vertices. If $g(\tilde{L}_0)$ and $g(\tilde{L}_\infty)$ have stumps, loops, or cut-vertices, then, by Theorem A, h-deg $Q(\tilde{L}_0)<c(\tilde{L}_0)-1$ and h-deg $Q(\tilde{L}_\infty)< c(\tilde{L}_\infty)-1$, respectively. If both $g(\tilde{L}_0)$ and $g(\tilde{L}_\infty)$ are connected graphs without stumps, loops and cut-vertices, by the hypothesis of induction the coefficient of the highest degree term is positive. Therefore h-deg $(x(Q(\tilde{L}_0)+Q(\tilde{L}_\infty)))=c(\tilde{L})-1$. Since $\tilde{L}_-$ is obtained from $\tilde{L}$ by switching a crossing, h-deg $Q(\tilde{L}_-) \leqq c(\tilde{L})-2$ by Lemma 2. Since $Q(\tilde{L}_+)=-Q(\tilde{L}_-)+x(Q(\tilde{L}_0)+Q(\tilde{L}_\infty))$ and the coefficient of the highest degree term is positive, we have that h-deg $Q(\tilde{L}_+)=c(\tilde{L}_+)-1$ and the coefficient of the highest degree term is positive. For the graph $g^*(\tilde{L})$, we can prove similarly and the proof is complete. $\square$

Proof of Corollary 2. If there exists $\tilde{L}$ such that $g(\tilde{L})$ and $g^*(\tilde{L})$ have no cut-vertices, then h-deg $Q(L)=c(\tilde{L})-1=c(L)-1$ by Theorem.

By Corollary 4 of [5] and Theorem A we have $c(L)-1=h$-deg $Q(L)=$ h-deg $Q(L_1)+h$-deg $Q(L_2) \leqq c(L_1)-1+c(L_2)-1=c(L_1)+c(L_2)-2=c(L)-2$. This is a contradiction. $\square$

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