47. Singularities of the Moduli Space of Yang-Mills Fields

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1. Let P be a G-bundle over a compact Kähler surface (M, h). We denote by $\mathcal{M}=\mathcal{M}(h)$ the set of all gauge equivalence classes of h-anti-self-dual (h-ASD) connections on P. The moduli space $\mathcal{M}(h)$ is a complex manifold possibly with singularities ([4], [6]). Actually around a point corresponding to a generic ASD connection $\mathcal{M}(h)$ has a complex manifold structure and around a point which is not generic it is described as either a complex analytic set or a stabilizer-quotient of a real analytic set. In this note we treat details of singular points on the moduli space.

2. For simplicity we assume that G is compact and semisimple. The base space M is assumed to be a compact, oriented Riemannian 4-manifold. An ASD connection A on P naturally induces the Yang-Mills complex: $\Omega^0(\mathfrak{g}_P)\xrightarrow{d_A}\Omega^1(\mathfrak{g}_P)\xrightarrow{d_A{}^+}\Omega^2_+(\mathfrak{g}_P)$. Here $\mathfrak{g}_P$ is the adjoint bundle $P\times_{\mathrm{Ad}}\mathfrak{g}$ ($\mathfrak{g}$ is the Lie algebra of G). The i-th cohomology group H^i_A of this complex is finite dimensional and the index $d=h^0-h^1+h^2$ is given by $\mathrm{Pont}_1(\mathfrak{g}^{\mathbf{C}}_P)+\dim G/2\times(\chi_M+\mathrm{sgn}_M)$, $h^i=\dim H^i_A$ ([3]). H^0_A is the Lie algebra of the stabilizer Γ_A, the group of gauge transformations of P fixing A. We call a connection generic when $H^0_A=0$ and $H^2_A=0$. The Yang-Mills complex presents completely in some sense information on local manifold structure of $\mathcal{M}(h)$ around the gauge equivalence class [A]. In fact $\mathcal{M}(h)$ has a Γ_A-quotient of a slice neighborhood $S_{A,\varepsilon}=\{\alpha\in\Omega^1(\mathfrak{g}_P),\ |\alpha|<\varepsilon,\ d^*_A\alpha=0,\ F_+(A+\alpha)=0\}$ and hence we have by making use of the Kuranishi map the following local structure theorem which was developed in for example [2], [11] in the case of $G=SU(2)$.

Theorem 1. *Singularities appear on $\mathcal{M}(h)$ exactly at ASD connections with $h^0\neq 0$ or with $h^2\neq 0$. Then $\mathcal{M}(h)$ admits possibly singularities of three types. Namely,* (i) *at a reducible ASD connection A with $h^2=0$ $\mathcal{M}(h)$ is locally homeomorphic to $\{x\in H^1_A\cong\boldsymbol{R}^{d+h^0};|x|<\varepsilon\}/\Gamma_A$ (quotient singularity),* (ii) *at A with $h^0=0$ and $h^2\neq 0$ there is an analytic map $\Psi;H^1_A\cong\boldsymbol{R}^{d+h^2}\to H^2_A\cong\boldsymbol{R}^{h^2}$ such that $\mathcal{M}(h)$ is locally homeomorphic around [A] to the zero point set $Zero(\Psi)_\varepsilon=\{x;\Psi(x)=0,|x|<\varepsilon\}$ (mapping critical point singularity) and* (iii) *at A with $h^0\neq 0$ and $h^2\neq 0$ there is a Γ_A-equivariant analytic map $\Psi;H^1_A\cong\boldsymbol{R}^{d'}\to H^2_A\cong\boldsymbol{R}^{h^2}$ $(d'=d+h^0+h^2)$ in such a way that $\mathcal{M}(h)$ is around [A] homeomorphic locally to the quotient of the zero point set $Zero(\Psi)_\varepsilon/\Gamma_A$ (composition of above two types).*

Remarks. The analytic maps Ψ are defined as $\Psi(\alpha)=\mathrm{pr}_{H^2}([f^{-1}\alpha\wedge f^{-1}\alpha]^+)$ where f is the Kuranishi map $\Omega^1(\mathfrak{g}_P)\to\Omega^1(\mathfrak{g}_P)$ and hence these can be approximated by quadratic maps. The stabilizer Γ_A acts as isometries on H^1_A with a canonical metric so that at a singular point [A] of type (i) $\mathcal{M}(h)$ has an orbifold structure provided that Γ_A fixes only the origin. H^0_A is independent of the Riemannian structure h in principle, whereas H^2_A depends on h. The holonomy group of A correctly determines Γ_A and hence H^0_A for Γ_A is the centralizer of the subgroup in G. $H^2_A=0$ if and only if the self-dual curvature map $\Omega^1(\mathfrak{g}_P)\to\Omega^2_+(\mathfrak{g}_P)$; $\alpha\mapsto F_+(A+\alpha)$ is surjective at A.

3. We assume that (M, h) is a compact Kähler surface. Then singularities of type (i) do not arise, since H^2_A is $\boldsymbol{R}$-isomorphic to $H^0_A\oplus \boldsymbol{H}^{0,2}$ ([6, Proposition 2.3]). Here $\boldsymbol{H}^{0,2}$ is the second cohomology group of the twisted Dolbeault complex: $\Omega^0(\mathfrak{g}_P^{\boldsymbol{C}})\xrightarrow{\bar\partial_A}\Omega^{0,1}(\mathfrak{g}_P^{\boldsymbol{C}})\xrightarrow{\bar\partial_A}\Omega^{0,2}(\mathfrak{g}_P^{\boldsymbol{C}})$. We note that cohomology groups $\boldsymbol{H}^0$, $\boldsymbol{H}^{0,1}$ and $\boldsymbol{H}^{0,2}$ are complex spaces, and $h^1=2h^{0,1}$ and $h^2=h^0+2h^{0,2}$ ($h^{0,i}=\dim_{\boldsymbol{C}}\boldsymbol{H}^{0,i}$). The last formula is a generalization of the well known formula $b^+=1+2p_g$.

Denote by $\mathcal{M}_{\rm gen}(h)$ the subset of $\mathcal{M}(h)$ of generic ASD connections on P. $\mathcal{M}_{\rm gen}(h)$ becomes a smooth manifold whose tangent space is H^1_A and H^1_A possesses a complex structure together with a Hermitian inner product induced naturally from the complex Kähler surface M. So $\mathcal{M}_{\rm gen}(h)$ is a complex manifold equipped with a Kähler structure ([7]). $\mathcal{M}(h)\backslash\mathcal{M}_{\rm gen}(h)$ consists exactly of all singular points and we have two types according to either case (a) in which A is irreducible ($h^0=0$) but $h^{0,2}\neq 0$ or case (b) in which A is reducible ($h^0\neq 0$).

Theorem 2. *Let P be a G-bundle over a compact Kähler surface (M, h).* (i) *At an ASD connection of type* (a) *$\mathcal{M}(h)$ is homeomorphic locally to a complex analytic set $Zero(\Psi)_\varepsilon=\{x\in \boldsymbol{H}^{0,1}; |x|<\varepsilon, \Psi(x)=0\}$, where Ψ; $\boldsymbol{H}^{0,1}\cong \boldsymbol{C}^{d''}\to \boldsymbol{H}^{0,2}\cong \boldsymbol{C}^{h^{0,2}}$, $d''=d/2+h^{0,2}$, is a holomorphic map.* (ii) *Around an ASD connection of type* (b) *$\mathcal{M}(h)$ is locally homeomorphic to the Γ_A-quotient of $Zero(\Psi)_\varepsilon=\{x\in H^1_A; |x|<\varepsilon, \Psi(x)=0\}$, where Ψ is a Γ_A-equivariant real analytic map $H^1_A\to H^2_A$.*

(ii) in Theorem 2 is just (iii) of Theorem 1, while (i) is verified because we get a local homeomorphism $S_{A,\varepsilon}\to S^{0,1}_{A,\varepsilon}$. Here $S^{0,1}_{A,\varepsilon}$ is a slice neighborhood of holomorphic connections modulo complex gauge transformations of P (see the discussion in § 4, [6]).

Remark. If (M, h) has positive total scalar curvature, then $\boldsymbol{H}^{0,2}_A=0$. We have further $\boldsymbol{H}^{0,2}_A=H^{0\boldsymbol{C}}_A$ over a surface with trivial canonical line bundle. The singularity of type (a) does not depend on any deformation of base Kähler metrics.

4. We investigate for the case $G=SU(2)$ the singularities of type (b) arising from reducible ASD connections. The following reduction theorem is mainly given in [5, Lemma 6.2].

Proposition 3. *Let P be an $SU(2)$-bundle over (M, h) with instanton number $c_2(P\times_\rho \boldsymbol{C}^2)=k$ ($\neq 0$). Then P admits a reducible ASD connection if and only if there is a holomorphic line bundle L with $c_1(L)^2=-k$ and satisfying $c_1(L)\wedge[\omega_h]=0$.*

$c_1(L) \wedge [\omega_h]$ is the obstruction for the existence of a Hermitian fibre metric on L whose curvature form is ASD. We remark that for an open dense set of Riemannian metrics on M, there are no complex line bundles with $U(1)$ ASD connections, since M has indefinite intersection form ([2, Corollary 3.21]).

As is well known, the set of $U(1)$-gauge equivalence classes of ASD connections on the $U(1)$-bundle L is parametrized by the Abelian variety $H^1_{deR}(M)/H^1(M; Z)$ of dimension b^1. For the set of gauge equivalence classes of reducible $SU(2)$-ASD connections on P, which we denote by $\mathcal{R}$, we have similarly the following.

Proposition 4 ([5, Lemma 6.5]). *Let $\{A_t\}$ be a one parameter family of reducible ASD connections on P with small $|t|$ which is non trivial with respect to gauge transformations. Then $\{A_t\}$ induces a harmonic 1-form as the infinitesimal deformation. Conversely each harmonic 1-form yields a one parameter family of reducible ASD connections.*

Since Γ_{A_0} acts trivially on the set $\left\{\begin{pmatrix} \sqrt{-1}a & 0 \\ 0 & -\sqrt{-1}a \end{pmatrix}; |a|<\varepsilon, a \text{ is a harmonic 1-form}\right\}$ which lies completely in $S_{A_0,\varepsilon}$, $\mathcal{R}$ is around $[A_0]$ a b^1 dimensional open ball. The number of connected components of $\mathcal{R}$ is counted by $l=1/2 \,\sharp\, \{L \in H^1(M; \mathcal{O}^*)$ with properties $c_1(L)^2=-k$, $c_1(L) \wedge [\omega_h]=0\}$. If $b^1=0$, singular points of type (b) appear in an isolated manner.

Define a hyperplane $c_1(L)^{\perp}$ in $H^2_{deR}(M)$ by $\{\alpha \in H^2_{deR}(M); \alpha \wedge c_1(L)=0\}$. Then $H^2_{deR}(M) \setminus \bigcup \{c_1(L)^{\perp}; L \in H^1(M; \mathcal{O}^*)$ with $c_1(L)^2=-k\}$ is not empty and we can deform the Kähler metric h to h_1 in its connected component so that from Proposition 3 the moduli space $\mathcal{M}(h_1)$ of h_1-ASD connections is a Kähler manifold possibly only with singular points of type (a).

5. Comments. Local structure theorems on $\mathcal{M}(h)$ are the same as the deformation theory of complex structures on a complex manifold except the stabilizer argument ([9]). We can give in the same way a local structure theorem on the moduli space of generalized ASD connections over a higher dimensional Kähler manifold (see for generalized ASD connection [8], [10]). Donaldson makes use of the reduction criterion similar to Proposition 3 to define new topological invariants on 4-manifolds and get a negative answer to Severi's question on rationality ([1]).

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