6. *Variations of Pseudoconvex Domains in the Complex Manifold*

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Introduction. In the n-dimensional complex vector space C^n with standard norm $\|z\|^2=|z_1|^2+\cdots+|z_n|^2$ for $z=(z_1,\cdots,z_n)\in C^n$, let D be a relatively compact domain of C^n with smooth boundary. Given $\zeta\in D$, D carries the Green's function $G(z)$ with pole at ζ for the Laplace equation $\Delta G=(\partial^2/\partial z_1\partial\bar{z}_1+\cdots+\partial^2/\partial z_n\partial\bar{z}_n)G=0$. The function $G(z)$ is expressed in the form

$$G(z)=\begin{cases}-\log|z-\zeta|+\lambda+H(z) & (n=1)\\ \|z-\zeta\|^{-2n+2}+\lambda+H(z) & (n\geqq 2)\end{cases}$$

where λ is a constant, $H(z)$ is harmonic in D and $H(\zeta)=0$. The constant term λ is called the Robin constant for $(D,\{\zeta\})$. When D varies in C^n with parameter t, so does λ with t. This is realized as follows: Let B be a domain of the t-complex plane containing the origin O. We let correspond to each $t\in B$ a relatively compact domain $D(t)$ of C^n with smooth boundary such that $D(t)\ni\zeta$ for all $t\in B$ and $D(O)=D$, and denote by $\lambda(t)$ the Robin constant for $(D(t),\{\zeta\})$. Consequently, $\lambda(t)$ defines a real-valued function on B. In [6] we showed

Theorem 1. *If the set $\tilde{D}=\{(t,z)\in B\times C^n\,|\,z\in D(t)\}$ is a pseudoconvex domain in $B\times C^n$, then $\lambda(t)$ is a superharmonic function on B.*

In this note we extend Theorem 1 to the case when $D(t)$ are domains in a complex manifold M.

1. Let M be a (compact or non-compact) connected complex manifold of dimension n. In this note we always assume that $n\geqq 2$, for we studied in [5] the case of $n=1$. Let $ds^2=\sum_{\alpha,\beta=1}^n g_{\alpha\beta}dz_\alpha\otimes d\bar{z}_\beta$ be a Hermitian metric on M. For notations we follow [3]. We put

$$\omega=i\sum_{\alpha,\beta=1}^n g_{\alpha\beta}dz_\alpha\wedge d\bar{z}_\beta,\qquad \omega^n=(i)^n\,n\,!\,g(z)dz_1\wedge d\bar{z}_1\wedge\cdots\wedge dz_n\wedge d\bar{z}_n,$$

$$\Delta=-(*\partial*\bar{\partial}+*\bar{\partial}*\partial)=-2\left\{\sum_{\alpha,\beta=1}^n g^{\alpha\beta}\frac{\partial^2}{\partial\bar{z}_\alpha\partial z_\beta}+\mathrm{Re}\sum_{\alpha,\beta=1}^n\frac{1}{g}\frac{\partial(gg^{\alpha\beta})}{\partial\bar{z}_\alpha}\frac{\partial}{\partial z_\beta}\right\},$$

where $i^2=-1$, $g(z)=\det(g_{\alpha\beta}(z))$ and $(g^{\alpha\beta}(z))=(g_{\alpha\beta}(z))^{-1}$. If a function u defined in a domain of M is of class C^2 and satisfies $\Delta u=0$, then u is said to be harmonic. For $\zeta\in M$ and a neighborhood U of ζ, we denote by $E(\zeta,U,ds^2)$ the set of all elementary solutions $E(\zeta,z)$ for $\Delta E(\zeta,z)=0$ on $U\times U$ except for the diagonal set (see K. Kodaira [2], p. 612).

In what follows, if M is compact, then we assume $D\neq M$. Moreover, we suppose $\zeta\in D$ and $E(\zeta,z)\in E(\zeta,U,ds^2)$.

First, consider the case where D is a relatively compact domain of M

with smooth boundary ∂D. Then D carries the Green's function $G(z)$ of D with pole at ζ which is uniquely determined by three conditions: G is harmonic in D except at ζ, $G(z)=0$ continuously on ∂D and $\lim_{z\to\zeta} G(z) \times r(z,\zeta)^{2n-2}=1$, where $r(z,\zeta)$ denotes the geodesic distance from z to ζ with respect to ds^2. Then G is expressed in a neighborhood of ζ in the form

$$G(z)=E(\zeta, z)+\lambda+H(z),$$

where λ is a constant, $H(z)$ is harmonic and $H(\zeta)=0$. The constant term λ is called the Robin constant for $(D, \{\zeta\})$ which corresponds to $E(\zeta, z)$.

Next, consider the case where D is a domain of M. Choose a sequence of relatively compact subdomains D_p $(p=1,2,\cdots)$ of D with smooth boundary such that $\zeta \in D_1$, $D_p \cup \partial D_p \subset D_{p+1}$ and $\bigcup_{p=1}^{\infty} D_p=D$. Each D_p carries the Green's function G_p with pole at ζ and the Robin constant λ_p for $(D_p, \{\zeta\})$ which corresponds to $E(\zeta, z)$. Since $G_p(z)$ and λ_p increase with p, the limits $G(z)=\lim_{p\to\infty} G_p(z)$ and $\lambda=\lim_{p\to\infty} \lambda_p$ exist, where it may happen that $G(z)\equiv +\infty$ on D, or equivalently $\lambda=+\infty$. We call G the Green's function of D with pole at ζ, and λ the Robin constant for $(D, \{\zeta\})$ which corresponds to $E(\zeta, z)$. As in the theory of Riemann surfaces ([1], Chap. IV), D with $\lambda=+\infty$ (resp. $<+\infty$) is said to be parabolic (resp. hyperbolic) for ds^2.

Finally, consider the case where D is an open set of M. When we denote by D_1 the connected component of D which contains ζ, we have the Green's function G_1 of D_1 with pole at ζ, and the Robin constant λ_1 for $(D_1, \{\zeta\})$ which corresponds to $E(\zeta, z)$. By the Green's function G of D with pole at ζ we mean $G=G_1$ on D_1 and $\equiv 0$ on $D-D_1$. By the Robin constant λ for $(D, \{\zeta\})$ which corresponds to $E(\zeta, z)$ we mean $\lambda=\lambda_1$.

Remark 1. In the special case where $M=\boldsymbol{C}^n$ and $ds^2=|dz_1|^2+\cdots+|dz_n|^2$, we have always $\lambda\leqq 0$. In [6], a domain D of $\boldsymbol{C}^n$ with $\lambda=0$ (resp. <0) was said to be parabolic (resp. hyperbolic).

2. Let M be a complex manifold with Hermitian metric ds^2, and let B be a domain of $\boldsymbol{C}$. Consider a domain $\tilde{D}$ of the product space $B\times M$ and put $D(t)=\tilde{D}\cap(\{t\}\times M)$ for $t\in B$, which is called a fiber of $\tilde{D}$ at t. As usual we can regard $\tilde{D}$ as variation of open set $D(t)$ of M with complex parameter $t\in B$. We write thus

$$\tilde{D}: t \longrightarrow D(t) \quad (t\in B).$$

Throughout this section we impose on $\tilde{D}$ the following conditions: (a) There exists a point $\zeta\in M$ such that $\tilde{D}\supset B\times\{\zeta\}$; (b) The boundary of $\tilde{D}$ in $B\times M$ is smooth; (c) Each $D(t)$ is a relatively compact domain of M with smooth boundary $\partial D(t)$. Now, take $E(\zeta, z)\in E(\zeta, U, ds^2)$. For any fixed $t\in B$ we have the Green's function $G(t, z)$ of $D(t)$ with pole at ζ and the Robin constant $\lambda(t)$ for $(D(t), \{\zeta\})$ which corresponds to $E(\zeta, z)$, so that G is expressed in a neighborhood of ζ in the form

$$G(t, z)=E(\zeta, z)+\lambda(t)+H(t, z) \tag{1}$$

where $H(t, z)$ is harmonic with respect to z and $H(t, \zeta)=0$. Consequently, $\lambda(t)$ becomes a function on B such that $-\infty<\lambda(t)<+\infty$. Since the variation $\tilde{D}\cup\partial\tilde{D}: t\to D(t)\cup\partial D(t)$ $(t\in B)$ is diffeomorphically trivial, $G(t, z)$ and

$\lambda(t)$ are of class C^3 on $(\tilde{D}\cup\partial\tilde{D})-B\times\{\zeta\}$ and on B, respectively. It follows from (1) that $\partial G/\partial t$ is of class C^2 on $\tilde{D}\cup\partial\tilde{D}$. In these circumstances we obtain the following fundamental inequality:

Theorem 2. *Suppose that $\tilde{D}$ is a pseudoconvex domain in $B\times M$. Then*

$$\frac{\partial^2\lambda(t)}{\partial t\partial\bar{t}}\leqq\frac{-2}{(n-1)\omega_{2n}}\left\{\left\|\bar{\partial}\frac{\partial G}{\partial t}\right\|^2_{D(t)}+\operatorname{Im}\iint_{D(t)}\left(\frac{\partial G}{\partial\bar{t}}\bar{\partial}\frac{\partial G}{\partial t}\wedge\partial*\omega+\frac{1}{2}\left|\frac{\partial G}{\partial t}\right|^2\bar{\partial}\partial*\omega\right)\right\}$$

for $t\in B$, where ω_{2n} is the surface area of the unit sphere in C^n.

3. Let us give some applications of Theorem 2. In this section except for 3°, we restrict ourselves to the same situation as in Theorem 2.

1° (Superharmonicity). Suppose that ds^2 satisfies the following condition: $\|\partial*\omega\|^2(z)\ \omega^n/n!\leqq\operatorname{Im}\bar{\partial}\partial*\omega$ on M, or equivalently

$$\sum_{\alpha,\beta=1}^{n}g^{\beta\alpha}\partial_{\bar{\beta}}T_{\alpha}\leqq 0\qquad\text{on } M,\tag{2}$$

where $T_\alpha=\sum_{\mu=1}^{n}T^{\mu}_{\mu\alpha}$ and $T^{\alpha}_{\mu\beta}=\Gamma^{\alpha}_{\mu\beta}-\Gamma^{\alpha}_{\beta\mu}$ (complex torsion). Then we obtain from Theorem 2

$$\frac{\partial^2\lambda(t)}{\partial t\partial\bar{t}}\leqq\frac{-1}{(n-1)\omega_{2n}}\left\|\bar{\partial}\frac{\partial G}{\partial t}\right\|^2_{D(t)}\qquad(\leqq 0).\tag{3}$$

Corollary 1. *If ds^2 satisfies* (2), *then $\lambda(t)$ is a superharmonic function on B.*

It is clear that any Koehler metric ds^2 on M satisfies (2). A simple example of non-Koehler metric satisfying (2) is $ds^2=\|dz\|^2/(1-\|z\|^2)^2$ on $M=\{z\in C^n\mid\|z\|<1\}$.

2° (Rigidity). By the inequality (3) we have

Lemma 1. *Suppose that ds^2 satisfies* (2). *Then,* (i) *if $(\partial^2\lambda/\partial t\partial\bar{t})(t_0)=0$ at some $t_0\in B$, then $(\partial G/\partial t)(t_0,z)=0$ on $D(t_0)$;* (ii) *if $\lambda(t)$ is harmonic on B, then D is identical with the product $B\times D(t_0)$.*

3° (Homogeneous spaces). Let M be a complex homogeneous (compact or non-compact) manifold with respect to a complex Lie transformation group G. Suppose that G admits a Koehler metric ds^2. Let D be a relatively compact pseudoconvex domain of M with non-empty smooth boundary. Construct the following subset of $G\times D$:

$$\tilde{D}=\{(g,z)\in G\times D\mid g(z)\in D\}.$$

Consequently, $\tilde{D}$ becomes a pseudoconvex open set of $G\times D$ and $\tilde{D}\supset\{e\}\times D$, where e is the unit element of G. We set $D(z)=\tilde{D}\cap(G\times\{z\})$ for $z\in D$. We regard $\tilde{D}$ as variation of open set $D(z)$ of G with parameter z of D, namely, $\tilde{D}:z\to D(z)$ $(z\in D)$. Choose $E(h,g)\in E(e,U,ds^2)$. For each $z\in D$, we consider the Robin constant $\lambda(z)$ for $(D(z),\{e\})$ which corresponds to $E(e,g)$. By Lemma 1

(i) $-\lambda(z)$ is a plurisubharmonic function on D such that

$$\lim_{z\to\partial D}(-\lambda(z))=+\infty;$$

(ii) if $-\lambda(z)$ is not strictly plurisubharmonic at some $z_0\in D$, then there exists a left invariant holomorphic vector field X on G such that $\{(\operatorname{Exp}tX)(g)(z_0)\mid t\in C\}$ is relatively compact in D (resp. ∂D, $M-(D\cup\partial D)$) for every $g\in G$ with $g(z_0)\in D$ (resp. ∂D, $M-(D\cup\partial D)$). Hence D never occurs to be a Stein manifold.

Remark 2. The assertion (ii) may be compared with the following theorem due to D. Michel [4]: In a compact homogeneous manifold with a complex Lie transformation group, any pseudoconvex domain which has at least one strictly pseudoconvex boundary point is a Stein manifold.

4. Let M, ds^2 and B be the same as in Section 2. Consider a domain $\tilde{D}$ of $B\times M$. Throughout this section, we suppose that (a) there exists a point $\zeta\in M$ such that $B\times\{\zeta\}\subset\tilde{D}$; (b) ds^2 is a Koehler metric on M; (c) $\tilde{D}$ admits a real analytic plurisubharmonic function φ on $\tilde{D}$ such that $\tilde{D}_r=\{\varphi<r\}$ is relatively compact in $\tilde{D}$ for any real r. Fix once and for all $E(\zeta, z)\in E(\zeta, U, ds^2)$ with ζ being the point mentioned in (a). For $t\in B$ we have the Robin constant $\lambda(t)$ for $(D(t), \{\zeta\})$ which corresponds to $E(\zeta, z)$. Thus $\lambda(t)$ is a function on B such that $-\infty<\lambda(t)\leqq+\infty$. Let B_0 be a relatively compact subdomain of B. Then there exists a real r_0 such that $\tilde{D}_r\supset B_0\times\{\zeta\}$ for all $r>r_0$. For each $t\in B_0$, we denote by $D_r(t)$ the fiber of $\tilde{D}_r$ at t, and have the Robin constant $\lambda_r(t)$ for $(D_r(t), \{\zeta\})$ which corresponds to $E(\zeta, z)$. In general, $\tilde{D}_r: t\to D_r(t)$ $(t\in B_0)$ is no longer diffeomorphically trivial, and hence $\lambda_r(t)$ is not always of class C^2 on B_0. However we shall find the following differentiability which is all we need:

Lemma 2. *For almost all r $(>r_0)$, $\lambda_r(t)$ is of class C^1 on B_0.*

This and Corollary 1 imply that for any such r, $\lambda_r(t)$ becomes superharmonic on B_0. Since $\lambda_r(t)\to\lambda(t)$ $(r\to+\infty)$ increasingly at $t\in B$, we have proved

Theorem 3. *The function $\lambda(t)$ is superharmonic on B.*

This yields the following fiber's uniformity:

Corollary 2. *Consider the subset $K=\{t\in B\,|\,D(t)$ has at least one parabolic connected component for $ds^2\}$. If K is of positive logarithmic capacity in $\mathbf{C}$, then $K=B$ and each connected component of $D(t)$ for every $t\in B$ is parabolic for ds^2.*

References

[1] L. V. Ahlfors and L. Sario: Riemann Surfaces. Princeton Univ. Press, N. J. (1960).

[2] K. Kodaira: Harmonic fields in Riemannian manifolds (generalized potential theory). Ann. of Math., **50**, 589–665 (1949).

[3] J. Morrow and K. Kodaira: Complex Manifolds. Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London (1971).

[4] D. Michel: Sur les ouverts pseudo-convexes des espaces homogènes. C. R. Acad. Sci. Paris, Sér. A, Math., **283**, 779–782 (1976).

[5] H. Yamaguchi: Parabolicité d'une fonction entière. J. Math. Kyoto Univ., **16**, 71–92 (1976).

[6] ——: Variations of pseudoconvex domains. Proc. Japan Acad., **59**A, 289–292 (1983).