46. *Degeneration of Kunev Surfaces. II*

By Sampei USUI*)
Department of Mathematics, Kochi University

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0. This note is a continuation of our previous report [13]. The essential result here is the determination of the main components of the whole degenerations of Kunev surfaces with finite local monodromy in the pure second cohomology (Theorem (3)). This is a progress for a compactification of the moduli space $\mathfrak{M}$ of Kunev surfaces. As a corollary, as in [13], this explains the relationship among the positive dimensional fibers of the period map in the pure second cohomology of Kunev surfaces and of elliptic surfaces with $p_g=1$ and $q=0, 1$ (Remark (2)).

A *Kunev surface* X is defined as a surface of general type with $\chi(\mathcal{O}_X)=2$ and with an involution σ such that X/σ is a K3 surface with rational double points (R.D.P., for short). This definition coincides with the one we adopted in [13] (cf. [1], [6], [12], [11]). We use the terminology a *homotopic K3 surface* and an *elliptic surface* as ones with $\kappa=1$ as before. We use the list of references in [13] freely as well as new ones added in the present article. Details will be published elsewhere.

1. A singularity of a reduced curve on a smooth surface is called *simple*, if the multiplicity is not bigger than three and if it is not an infinitely near triple point. This is equivalent to say that the double cover of the surface branched along the curve has R.D.P. For sextic curves on $\boldsymbol{P}^2$, curves with at most simple singularities coincide with properly stable curves with respect to the action of SL_3 ([3], [5]). Set $\mathfrak{N}=\{\sum C_j\in \mathrm{Sym}^2|\mathcal{O}_{\boldsymbol{P}^2}(3)|\ |\sum C_j$ has only simple singularities$\}/SL_3$. Then $\mathfrak{N}$ can be seen as the coarse moduli space of the K3 surfaces with R.D.P. which are quotients of Kunev surfaces X by their involution σ, and we have a map $\pi:\mathfrak{M}\to\mathfrak{N}$, $[X]\mapsto[X/\sigma]$.

2. For any fixed $[\sum C_j]\in\mathfrak{N}$, we define functions in $t\in\check{\boldsymbol{P}}^2$ by

$$m(t)=\textstyle\sum_{P\in\boldsymbol{P}^2}\min\{I(P, L_t\cap C_j)\,|\,j=1,2\},\quad\text{and}$$
$$n(t)=\#\{\text{triple points of } C_j \text{ on } L_t,\ j=1,2\}.$$

Notice that if C_j has a triple point then C_j consists of three distinct lines with a common point. These functions define two stratifications of $\check{\boldsymbol{P}}^2$:

$$\check{\boldsymbol{P}}^2=S_0\cup S_1\cup S_2,\qquad \text{where } S_m=\{t\in\check{\boldsymbol{P}}^2\,|\,m=\min\{2, m(t)\}\}.$$
$$\check{\boldsymbol{P}}^2=T_0\cup T_1\cup T_2,\qquad \text{where } T_n=\{t\in\check{\boldsymbol{P}}^2\,|\,n=n(t)\}.$$

Notice that codim $S_m=m$, codim $T_0=0$, and codim $T_n=n$ if T_n is non-empty $(n=1, 2)$.

3. For $[\sum C_j]\in\mathfrak{N}$, we denote by Y the minimal K3 surface which is

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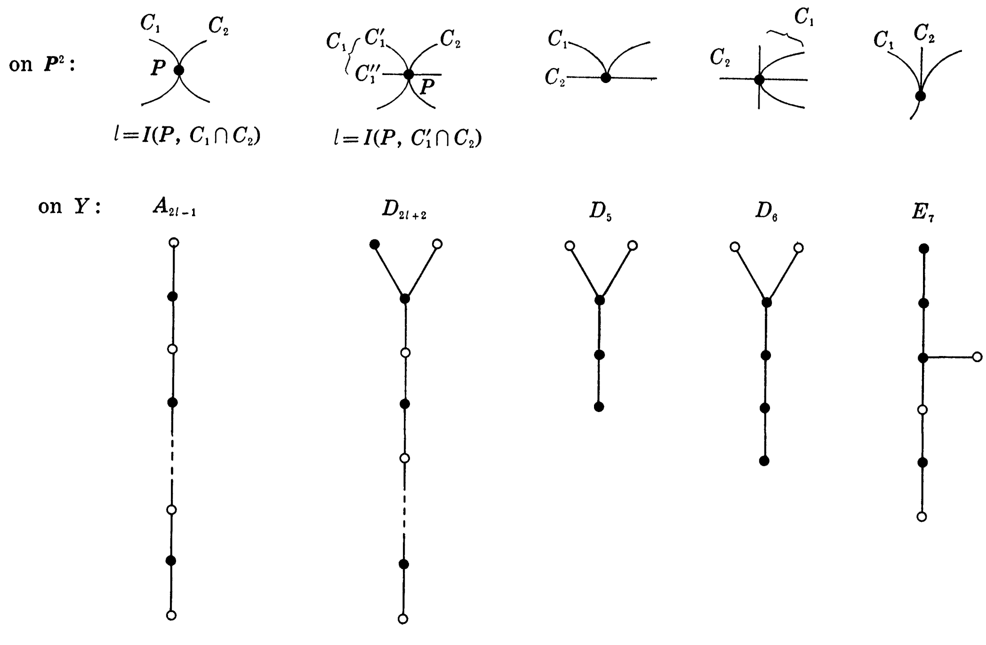
obtained as the minimal resolution of the double cover of $\boldsymbol{P}^2$ branched along $\sum C_j$. Let $\alpha_1 : Y \to \boldsymbol{P}^2$ be the projection and E_i be the exceptional curves for α_1, i.e., (-2)-curves. Then we have the following lemma:

Lemma. *The sets* $\{E_i \mid$ *the multiplicity of* E_i *in the total transform of* C_j *is odd*$\}$ $(j=1, 2)$ *coincide and the number of their element is nine.*

Remark. The nine (-2)-curves in the above lemma is an equivalent datum to the one of the *distinguished partial desingularization* of a K3 surface of Kunev (more generally, Todorov) type in [11]. We call the former the *distinguished* (-2)-*curves*. They appeared in A.D.E. configuration of exceptional curves over R.D.P. in the following way:

Table



o: distinguished (-2)-curve

We reorder the numbering so that E_i $(1 \le i \le 9)$ are the nine distinguished (-2)-curves on Y, and set $\mathcal{E}_i = \check{\boldsymbol{P}}^2 \times E_i$ $(1 \le i \le 9)$. Denote by $\mathcal{L} \subset \check{\boldsymbol{P}}^2 \times \boldsymbol{P}^2$ the total space of the universal family of lines on $\boldsymbol{P}^2$. We can construct families of surfaces $f : \mathcal{X} \to \check{\boldsymbol{P}}^2$ and $\tilde{f} : \tilde{\mathcal{X}} \to \check{\boldsymbol{P}}^2$ in the following way: (0) Set $\alpha = 1 \times \alpha_1 : \check{\boldsymbol{P}}^2 \times Y \to \check{\boldsymbol{P}}^2 \times \boldsymbol{P}^2$. (i) Let $\beta : \mathcal{Y} \to \check{\boldsymbol{P}}^2 \times Y$ be the blowing-up along $\alpha^{-1}\mathcal{L} \cap (\sum_{1 \le i \le 9} \mathcal{E}_i)$. Denote by $\mathcal{W}_i$ $(1 \le i \le 9)$ the exceptional divisors. (ii) Take the double cover $\gamma : \tilde{\mathcal{X}}' \to \mathcal{Y}$ branched along $(\alpha\beta)^{-1}\mathcal{L} + \beta^{-1}(\sum \mathcal{E}_i)$. (iii) Let $\delta : \tilde{\mathcal{X}}' \to \tilde{\mathcal{X}}$ be the contraction of $(\beta\gamma)^{-1}(\sum \mathcal{E}_i)$. (iv) Let $\varepsilon : \tilde{\mathcal{X}} \to \mathcal{X}$ be the contraction of $\delta(\beta\gamma)^{-1}(\sum \mathcal{W}_i)$. (In the notation above, we used $\alpha^{-1}\mathcal{L}$ etc. as the proper transforms.)

Set $\mathcal{L}_{\tilde{\mathcal{X}}} = (\delta(\alpha\beta\gamma)^{-1}\mathcal{L}$ with reduced structure) and $\mathcal{W}_{\tilde{\mathcal{X}},i} = \delta\gamma^{-1}\mathcal{W}_i$.

4. Theorem. *With the above notation,* $f : \mathcal{X} \to \check{\boldsymbol{P}}^2$ *is a complete family* *of degenerations of Kunev surfaces over the fixed* $[\sum C_j] \in \mathfrak{N}$. *This family has the following properties*:

(1) *The singularity of the total space* $\mathcal{X}$ *consists of disjoint nine compounds Veronese cone over* $S_1 \cup S_2 = (C_1 \cap C_2)^\vee$ *(for the terminology, see* [13], *for example).* $\varepsilon: \tilde{\mathcal{X}} \to \mathcal{X}$ *is a desingularization and the exceptional divisor* $\mathcal{W}_{\tilde{\mathcal{X}},i}$ *is a family of* $\boldsymbol{P}^2$ *over a line in* $(C_1 \cap C_2)^\vee$ $(1 \leq i \leq 9)$. $K_{\tilde{\mathcal{X}}} = \mathcal{L}_{\tilde{\mathcal{X}}} + \sum \mathcal{W}_{\tilde{\mathcal{X}},i}$.

(2) *The fiber* $\tilde{X}_t := \tilde{f}^{-1}(t) = V_t + \sum W_{i,t}$, *where* V_t *is the main component and* $W_{i,t} := \mathcal{W}_{\tilde{\mathcal{X}},i} | \tilde{X}_t$. *Hence the dualizing sheaf of* V_t *coincides with* $\mathcal{O}(\mathcal{L}_{\tilde{\mathcal{X}}} | V_t)$.

(3) V_t *is a (singular) Kunev surface, homotopic K3 surface, K3 surface, elliptic surface with* $p_g = q = 1$, *or abelian surface according to* $t \in S_0 \cap T_0$, S_1, S_2, $S_0 \cap T_1$, *or* T_2.

Remark. (1) We have described series of degenerations of the canonical divisor K_t of the minimal model of V_t for $t \in S_1$ and $t \in S_0 \cap T_1$, but because of the limit of pages we only point out the following: (i) In case $t \in S_1$, K_t is the reduced curve of type ${}_2I_{2l}$ $(0 \leq l \leq 8)$ in Kodaira's notation of singular fibers of elliptic fibrations [10]. (ii) In case $t \in S_0 \cap T_1$, K_t is I_{2l} $(0 \leq l \leq 4)$, IV, I_0^*, or IV^*.

(2) As Corollary 3 in [13], (3) in Theorem says that $S_0 \cap T_0$, S_1, and $S_0 \cap T_1$ appear as positive dimensional fibers of the period map of the pure second cohomology of Kunev surfaces, homotopic K3 surfaces, and elliptic surfaces with $p_g = q = 1$, respectively (cf. [7], [8], [9], [4], [6]).

(3) Since the isotropy group $\mathrm{Isot}(\sum C_j)$ in SL_3 is finite for any $\sum C_j \subset \boldsymbol{P}^2$ with $[\sum C_j] \in \mathfrak{N}$, $\check{\boldsymbol{P}}^2 / \mathrm{Isot}(\sum C_j)$ is actually a completion of the fiber $\pi^{-1}[\sum C_j]$.

Idea of Proof of Theorem (3). Consider the reduced subcycle $B_{Y,t}$ of $\alpha_1^* L_t +$ (nine distinguished (-2)-curves) on Y consisting of those components whose multiplicities are odd. Then the double cover of Y branched along $B_{Y,t}$ is the normalization of the main component V_t. By local classification of the possibilities of configurations of two cubics $\sum C_j$ and a line L_t, together with the global restrictions subjected by Bezout's Theorem, we can list up all the possibilities of $B_{Y,t}$. From this, we can calculate the canonical divisors K_t of the minimal models of V_t. We also use the elliptic fibrations induced from pencils of lines on $\boldsymbol{P}^2$ passing a common point in $C_1 \cap C_2$ in case $t \in S_1$ or a triple point of C_j in case $t \in T_1$.

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