45. An Elementary Variant of Nonstandard Set Theory

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The purpose of this paper is to provide a formal system $*$NST, which is more elementary than the system UNST ([1]). We note that the set U of all "usual sets" need not be a model for ZFC. The set U will be described explicitly and will be a model for ZC, where ZC means ZFC minus replacement axiom. It is well known that almost all of modern mathematics can be developed within ZC.

The basic symbols of the language of $*$NST are $\in$, I and $*$. The symbol $\in$ denotes membership. I and $*$ are constant symbols. Various letters $x, y, \cdots, X, Y, \cdots, \mathcal{F}, \cdots$ are variables. The formula "$x \in I$" means that x is an internal set. We shall use the word "set" for "external set" in [1], and define external set to be non-internal set. If the free variables of a formula ϕ in ZFC or in $*$NST are among $x_1, \cdots, x_n$, then we write as

$$\phi(x_1, \cdots, x_n).$$

We adopt for the universe of all sets the axioms of ZFC$^-$, i.e. the axioms of ZFC minus the axiom of foundation, for a while. So we can use freely definitions and theorems that can be developed set theoretically with ZFC$^-$ ([2]). Especially we use boldface letters for classes. Let $\boldsymbol{V}$ be the class of all sets and $\boldsymbol{O}$ be the class of all ordinals. Define as usual $R(\alpha)$ for $\alpha \in \boldsymbol{O}$, and let $\boldsymbol{W} = \bigcup \{R(\alpha) \mid \alpha \in \boldsymbol{O}\}$. Then the class $\boldsymbol{W}$ is a transitive model for ZFC. Let ω be the set of all natural numbers. If ξ is a limit ordinal$>\omega$, and if we let $U = R(\xi)$, then the set U is a transitive model for ZC.

Definition. (a) A set $\mathcal{F}$ is called of U-size, if there is a surjection from U onto $\mathcal{F}$.

(b) A set $\mathcal{F}$ is finite, if there is a natural number $n \in \omega$ and there is a bijection from n onto $\mathcal{F}$.

Now the axioms of $*$NST are the following:

Axiom 1. ZFC$^-$ axioms for formulas of $*$NST.

Axiom 2 (Nonstandard analysis).

(a) The set I is transitive: $x \in I \Rightarrow x \subseteq I$.

(b) There exists a limit ordinal $\xi > \omega$ such that

$$* : R(\xi) \longrightarrow I.$$

Let $U = R(\xi)$; we use $^*x$ instead of $*(x)$ for $x \in U$. As noted above, the

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set U is a transitive model for ZC.

(c) (Transfer principle). Let $\phi\ (x_1, \cdots, x_n)$ be a formula in ZFC and $x_1, \cdots, x_n$ be in U. Then

$$\phi^U(x_1, \cdots, x_n) \Longleftrightarrow \phi^I({}^*x_1, \cdots, {}^*x_n);$$

where ϕ^U (resp. ϕ^I) is the relativisation of ϕ to U (resp. I).

(d) (Saturation principle). Let $\phi\ (x, X, w_1, \cdots, w_n)$ be a formula in ZFC, $w_1, \cdots, w_n$ be in I, and $\mathcal{F}$ be a set of U-size, contained in I. Then

$$\forall F: \text{finite} \subseteq \mathcal{F} \quad \exists x \in I \quad \forall X \in F \quad \phi^I\ (x, X, w_1, \cdots, w_n)$$
$$\Longrightarrow \exists x \in I \quad \forall X \in \mathcal{F} \quad \phi^I\ (x, X, w_1, \cdots, w_n).$$

Axiom 3. (Foundation over I). $X \neq 0 \wedge X \cap I = 0 \Rightarrow X$ has an $\in$-minimal element; where 0 is the empty set.

By using these axioms we have the following results.

Proposition 1. $\forall x (x \notin x)$; $\forall x, y (\neg (x \in y \wedge y \in x))$.

Proof. From Axiom 2(c) the set I is also a transitive model for ZC. Axiom 3 implies these results.

Proposition 2. (a) $U \cap I = W \cap I = R(\omega)$.

(b) *For* $x \in U$, ${}^*x = x \Leftrightarrow x \in R(\omega)$.

Proof. Every set $A \subseteq W$ has an $\in$-minimal element if $A \neq 0$. We have the result (a) by using countable saturation priciple.

For $x \in U$ the function $x \to {}^*x$ from X into ${}^*X$ is injective by transfer principle. Therefore, if we are not concerned with the structure of the set x, we can consider X as a subset of ${}^*X$ by identifying ${}^*x$ with x. Thus we can develop nonstandard analysis.

Now we define the set $S(\alpha)$ for $\alpha \in \boldsymbol{O}$ by:

(a) $S(0) = I$,

(b) $S(\alpha+1) = \{x \mid x \subseteq S(\alpha)\}$,

(c) $S(\alpha) = \cup \{S(\gamma) \mid \gamma < \alpha\}$ when α is a limit ordinal.

Let $\boldsymbol{S} = \cup \{S(\alpha) \mid \alpha \in \boldsymbol{O}\}$, then $\boldsymbol{S} \subseteq \boldsymbol{V}$. We have the following.

Proposition 3. (a) *For* $\alpha \in \boldsymbol{O}$, $S(\alpha) \cap W = R(\omega + \alpha)$.

(b) *Axiom* 3 *is equivalent to* $\boldsymbol{S} = \boldsymbol{V}$.

Proof. We prove (b) by defining $\text{rank}_I(x)$ for $x \in \boldsymbol{S} - I$, which is analogous to $\text{rank}(x)$ for $x \in W$.

Proposition 4. *$*$NST has a model in ZFC.*

Proof. We follow the method of [1] and construct the Mostowski collapse function in the model, and use its inverse function as an interpretation of the constant symbol $*$.

References

[1] T. Kawai: Nonstandard analysis by axiomatic method. Southeast Asian Conference on Logic. Elsevier Science Publishers B. V., North Holland, (eds. C. T. Chong and M. J. Wicks), pp. 55–76 (1983).

[2] K. Kunen: Set Theory. An Introduction to Independence Proofs, North Holland (1980).