42. Finite Multiplicity Theorems for Induced Representations of Semisimple Lie Groups and their Applications to Generalized Gelfand-Graev Representations

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Let G be a connected real semisimple Lie group with finite center, and $\mathfrak{g}$ its Lie algebra. *Generalized Gelfand-Graev representations* (GGGRs), first introduced by Kawanaka [2] for finite reductive groups, form a series of induced representations of G parametrized by nilpotent $\mathrm{Ad}(G)$-orbits in $\mathfrak{g}$. In particular, a principal nilpotent class gives rise to a representation of G induced from a non-degenerate character of a maximal unipotent subgroup. This special type of GGGR, attributed to Gelfand-Graev, is of multiplicity free if G is quasi-split [5].

In this note, we first generalize van den Ban's finite multiplicity theorem [1] for the quasi-regular representation $\mathrm{Ind}_H^G(1_H)$ associated with a semisimple symmetric space G/H, and give nice sufficient conditions for induced representations of G to be of multiplicity finite. Then, applying these criterions, we show that certain interesting types of GGGRs, closely related to the regular representation of G, have finite multiplicity property. Our finite multiplicity theorems are given for *reduced* GGGRs (RGGGRs), a variant of GGGRs. We also give a multiplicity one theorem for RGGGRs under some additional assumptions.

1. Criterions for finite multiplicity property. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, and θ the corresponding Cartan involution of $\mathfrak{g}$, which can be lifted up canonically to an involution of G. Denote by K the maximal compact subgroup of G consisting of fixed points of θ on G. Let $Q=LN$ with $L\equiv Q\cap\theta Q$, denote a Levi decomposition of a parabolic subgroup Q of G. Let σ be an involutive automorphism of $\mathfrak{l}\equiv\mathrm{Lie}(L)$ satisfying: (1) σ commutes with $\theta|\mathfrak{l}$, and (2) σ coincides with θ on the split component $\mathfrak{a}$ of $\mathfrak{l}$. Take a closed subgroup H of L with Lie algebra $\mathfrak{h}\equiv\{X\in\mathfrak{l}\,;\,\sigma X=X\}$.

For a continuous representation ζ of the semidirect product subgroup $HN=H\ltimes N$ on a Fréchet space $\mathcal{F}$, we consider the representation C^∞-$\mathrm{Ind}_{HN}^G(\zeta)=(\pi_\zeta, C^\infty(G\,;\zeta))$ of G induced from ζ in C^∞-context: the group G acts on the representation space

$$C^\infty(G\,;\zeta)\equiv\{f\colon G\xrightarrow{C^\infty}\mathcal{F}\,;\, f(gz)=\zeta(z)^{-1}f(g)\ (g\in G,\, z\in HN)\},$$

by left translation. $C^\infty(G\,;\zeta)$ has a $U(\mathfrak{g}_C)$-module structure through differentiation, where $U(\mathfrak{g}_C)$ denotes the enveloping algebra of $\mathfrak{g}_C\equiv\mathfrak{g}\otimes_{\boldsymbol{R}}\boldsymbol{C}$. Let $\mathfrak{Z}$ be the center of $U(\mathfrak{g}_C)$. For an algebra homomorphism $\chi\colon\mathfrak{Z}\to\boldsymbol{C}$, the joint

χ-eigenspace $C^\infty(G;\zeta:\chi)$ for $\pi_\zeta(\mathfrak{Z})$ is G- and $\mathfrak{g}_C$-stable. For $\tau\in\hat{K}$, the unitary dual of K, denote by $\mathcal{A}(G;\zeta:\chi)_\tau$ the τ-isotypic component of $C^\infty(G;\zeta:\chi)$. An element of $\mathcal{A}(G;\zeta:\chi)_\tau$, necessarily real analytic on G, is called a $(\tau,\zeta:\chi)$-spherical function.

Now let ρ be an irreducible admissible $(\mathfrak{g}_C, K)$-module with infinitesimal character χ. Every embedding of ρ into π_ζ carries τ-isotypic vectors for ρ to $(\tau,\zeta:\chi)$-spherical functions. So, the multiplicity $I(\rho,\pi_\zeta)$ of ρ in π_ζ as submodules is bounded as

(1.1) $$I(\rho,\pi_\zeta)\leqq \min_{\tau\in\hat{K}}[(\dim\mathcal{A}(G;\zeta:\chi)_\tau)/I_K(\tau,\rho)],$$

where $I_K(\tau,\rho)$ denotes the multiplicity of τ in ρ as a K-module.

Suggested by this inequality, we estimate dimensions of spaces $\mathcal{A}(G:\zeta:\chi)_\tau$ of spherical functions. Let $\mathfrak{q}$ be the (-1)-eigenspace for σ on $\mathfrak{l}$, and $\mathfrak{a}_{pq}(\supseteqq\mathfrak{a})$ a maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$. The centralizer $\mathfrak{l}'$ of $\mathfrak{a}_{pq}$ in $\mathfrak{g}$ is a reductive Lie subalgebra of $\mathfrak{g}$. Denote by M_{kh} the centralizer of $\mathfrak{a}_{pq}$ in $K\cap H$.

Theorem 1. *Let R_1 and R_2 denote the orders of Weyl groups of $\mathfrak{g}_C$ and $\mathfrak{l}'_C$ respectively. Then one has*

$$\dim\mathcal{A}(G;\zeta:\chi)_\tau\leqq(R_1/R_2)\cdot\dim\tau\cdot I_{M_{kh}}(\tau,\zeta),$$

where $I_{M_{kh}}(\tau,\zeta)$ denotes the intertwining number from $\tau|M_{kh}$ to $\zeta|M_{kh}$.

This theorem together with (1.1) yields in particular a hereditary character of finite multiplicity property as follows.

Theorem 2. *The induced representation $\pi_\zeta=C^\infty\text{-}\mathrm{Ind}_{HN}^G(\zeta)$ has finite multiplicity property if so does the restriction of ζ to M_{kh}.*

Now assume ζ to be unitary, and consider the unitarily induced representation $U_\zeta\equiv L^2\text{-}\mathrm{Ind}_{HN}^G(\zeta)$. Let $U_\zeta\simeq\int_{\hat{G}}^{\oplus}[m_\zeta(\beta)]\cdot\beta d\mu_\zeta(\beta)$ be the factor decomposition of U_ζ, where μ_ζ is a Borel measure on the unitary dual $\hat{G}$ of G, and $m_\zeta:\ \hat{G}\to\{0,1,2,\cdots\}\cup\{\infty\}$, the multiplicity function for U_ζ. Using the result of Penney [4], we can show that

(1.2) $$m_\zeta(\beta)\leqq I(\rho_\beta,\pi_\zeta)\quad\text{for almost all }\beta\in\hat{G}\text{ with respect to }\mu_\zeta,$$

at least when ζ is finite-dimensional. Here, ρ_β denotes the irreducible $(\mathfrak{g}_C, K)$-module of K-finite vectors for a unitary representation of class $\beta\in\hat{G}$. Thus one obtains

Theorem 3. *The L^2-induced representation U_ζ is of multiplicity finite whenever ζ is finite-dimensional.*

These two theorems cover various finite multiplicity theorems for induced representations of G, especially the case of van den Ban as $Q=L=G$, $\zeta=1_H$, the trivial character of H.

2. GGGRs Γ_i. Hereafter, assume that G/K is an irreducible hermitian symmetric space. We construct the (reduced) GGGRs explicitly. (See [2], [6] for the definition of GGGRs in full generality.)

Let $G=KA_pN_m$ be an Iwasawa decomposition of G. Put $l=\dim A_p$. By Moore [3], the Dynkin diagram of the root system of $(\mathfrak{g},\mathfrak{a}_p)$, $\mathfrak{a}_p\equiv\mathrm{Lie}(A_p)$, is expressed as

(I) $\overset{\alpha_1}{\circ}$——$\overset{\alpha_2}{\circ}$—$\cdots$—$\overset{\alpha_{l-1}}{\circ}\Leftarrow\overset{\alpha_l}{\circ}$ if G/K is of tube type,

(II) $\overset{\alpha_1}{\circ}$——$\overset{\alpha_2}{\circ}$—$\cdots$—$\overset{\alpha_{l-1}}{\circ}\Rightarrow\overset{\alpha_l}{\circ}$ if G/K is of non-tube type.

Let $Q=LN$, $Q\supseteqq MA_pN_m$, be the maximal parabolic subgroup of G such that $(\alpha_j)_{1\leqq j<l}$ generates the root system of its Levi subgroup L, where M denotes the centralizer of A_p in K. Then the unipotent radical N is an at most two-step nilpotent Lie subgroup of N_m, canonically diffeomorphic to the Šilov boundary of Siegel domain realizing G/K. N is abelian exactly in the above case (I). The Levi component L acts on $\mathfrak{n}\equiv\mathrm{Lie}(N)$ and on the center $\mathfrak{z}_\mathfrak{n}$ of $\mathfrak{n}$, through the adjoint action.

Proposition 4. (1) *The center $\mathfrak{z}_\mathfrak{n}$ admits precisely $(l+1)$-number of open* $\mathrm{Ad}(L)$*-orbits $\tilde{\omega}_i$ $(0\leqq i\leqq l)$, numbered as $\tilde{\omega}_i=-\tilde{\omega}_{l-i}$.*

(2) *The nilpotent* $\mathrm{Ad}(G)$*-orbits $\omega_i\equiv\mathrm{Ad}(G)\tilde{\omega}_i$, $0\leqq i\leqq l$, are all contained in the same nilpotent class o of $\mathfrak{g}_C$,*

(3) *$o\cap\mathfrak{g}$ splits as $o\cap\mathfrak{g}=\coprod_{0\leqq i\leqq l}\omega_i$ (disjoint union).*

For any fixed i, take an element X from $\tilde{\omega}_i$, and define a linear form X^* on $\mathfrak{n}$ by $\langle X^*, Z\rangle=B(Z,\theta X)$ $(Z\in\mathfrak{n})$, where B is the Killing form of $\mathfrak{g}$. Then, there exists a unique (up to equivalence) irreducible unitary representation ξ_X of N for which the center $Z_N=\exp\mathfrak{z}_\mathfrak{n}$ of N is represented by scalars: $Z_N\ni\exp Z\longmapsto\exp\sqrt{-1}\langle X^*, Z\rangle$. This representation ξ_X is one-dimensional or infinite-dimensional according as the case (I) or (II). We put $\Gamma_i=(C^\infty$- or L^2-$)\,\mathrm{Ind}_N^G(\xi_X)$. Then, the equivalence class of Γ_i does not depend on the choice of an $X\in\tilde{\omega}_i$ because $\tilde{\omega}_i$ is a single $\mathrm{Ad}(L)$-orbit. The induced representation Γ_i is called the GGGR associated with ω_i.

Our unitary GGGRs L^2-Γ_i are closely related to the regular representation $(\lambda_G, L^2(G))$ of G as follows.

Proposition 5. *One has $\lambda_G\simeq\oplus_{0\leqq i\leqq l}[\infty]\cdot L^2$-$\Gamma_i$ (unitary equivalence). So, any discrete series of G is embedded into GGGR L^2-Γ_i for some i.*

We give in [7] a complete description of embeddings, or Whittaker models, of holomorphic discrete series into C^∞- or L^2-GGGRs Γ_i.

3. RGGGRs $\Gamma_i(c)$. We now fix an element $A[i]\in\tilde{\omega}_i$, and put $\xi_i=\xi_{A[i]}$. Since L acts on N, it acts also on the unitary dual $\hat{N}$ of N in the canonical way. Let H^i be the stabilizer of the equivalence class of ξ_i in L. Then H^i is reductive. We can show that ξ_i is extendable to an actual (not just projective) unitary representation $\tilde{\xi}_i$ of the semidirect product subgroup H^iN acting on the same Hilbert space. For an irreducible (unitary, in case of L^2-Ind) representation c of H^i, the induced representation $\Gamma_i(c)$ $\equiv\mathrm{Ind}_{H^iN}^G(\tilde{c}\otimes\tilde{\xi}_i)$ with $\tilde{c}\equiv c\otimes 1_N$, is called the RGGGR associated with (ω_i, c). The GGGR L^2-Γ_i is decomposed into a direct integral of RGGGRs L^2-$\Gamma_i(c)$ $(c\in(H^i)^\wedge)$.

4. Finite multiplicity theorems for RGGGRs. We remark that (L, H^i) has, for every i, a structure of reductive symmetric pair (on Lie algebra level) attached to a signature of the root system of L. Thus one can apply Theorems 2 and 3 to RGGGRs $\Gamma_i(c)$. Using Fock model realization of $\tilde{\xi}_i$ (see [7]), we examine multiplicities in $\tilde{\xi}_i | M_{kh}$ with $M_{kh} = M$, in detail. Consequently, we get the following

Theorem 6. *The C^∞-induced RGGGR C^∞-$\Gamma_i(c)$ is of multiplicity finite whenever c is finite-dimensional.*

There exist precisely two cases: (say) $i=0$, l, for which the stabilizer H^i is a maximal compact subgroup of L.

Theorem 7. (1) *If $i=0$ or l, all the unitary RGGGRs L^2-$\Gamma_i(c)$ coming from Γ_i have finite multiplicity property.*

(2) *If G/K is of tube type, L^2-$\Gamma_i(c)$ is of multiplicity finite for any i and any finite-dimensional c.*

Remark. Theorem 7(1) can not be obtained directly from Theorem 3 because $\dim \xi_i = \infty$ in general. Nevertheless, the estimation (1.2) still holds in the present case. So we get Theorem 7(1) from Theorem 6.

5. A multiplicity one theorem. Assume that G be linear and G/K of tube type. By generalizing the technique in [5], [6], we can prove

Theorem 8. *Let $i=0$ or l. Take as an extension $\tilde{\xi}_i$ of ξ_i a unitary character of H^iN trivial on H^i. If c is a real-valued character of the maximal compact subgroup H^i of L, then the unitary RGGGR L^2-$\Gamma_i(c)$ $= L^2$-$\mathrm{Ind}_{H^iN}^G (\check{c} \otimes \tilde{\xi}_i)$ is of multiplicity free.*

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