36. *On Some Points in Vector Analysis*

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1. In the electro-magnetic theory of James Clerk Maxwell, one encounters the notions of intensities and of fluxes. Both are represented by triples of real numbers referred to a system of orthonormal coordinates or a frame $(O; \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ in the Euclidean 3-space. When this frame is transformed to another frame $(O'; \boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3)$, then $(\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3)$ should be $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)T$ with an orthogonal matrix T. If an intensity is represented by (x_1, x_2, x_3) referred to $(O; \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ then it will be represented by $(x'_1, x'_2, x'_3) = (x_1, x_2, x_3)T$ when referred to $(O'; \boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3)$, whereas a flux represented by (x_1, x_2, x_3) in the first frame will be represented by $(x_1, x_2, x_3)(\det T)T$ in the second frame. In the literature, the *intensity* in this sense is often called the *vector*, and the *flux* is called the *pseudo-vector*. Analogously, the *pseudo-scalar* is often defined as the "quantity" represented by a real number x with respect to a frame, x being replaced by $x(\det T)$ when the frame is transformed.

The purpose of this paper is to give mathematical definitions of pseudo-vectors and of pseudo-scalars and to show their usefulness in clarifying some points in geometry, kinematics and electro-magnetic theory in the Euclidean 3-space.

2. Let $\boldsymbol{E}$ be the Euclidean 3-space with the vector space $\boldsymbol{V}$, and $\wedge^p(\boldsymbol{V})$ be a p-fold exterior power of $\boldsymbol{V}$, $p=1,2,3$. Then $\dim \wedge^2(\boldsymbol{V})=3$, $\dim \wedge^3(\boldsymbol{V})=1$. Let $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$, $(\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3)$ be two sets of orthonormal bases of $\boldsymbol{V}$, $(\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3) = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)T$ with an orthogonal matrix T, then $\wedge^2(\boldsymbol{V})$ has two sets of bases $(\boldsymbol{e}_2\wedge\boldsymbol{e}_3, \boldsymbol{e}_3\wedge\boldsymbol{e}_1, \boldsymbol{e}_1\wedge\boldsymbol{e}_2)$, $(\boldsymbol{e}'_2\wedge\boldsymbol{e}'_3, \boldsymbol{e}'_3\wedge\boldsymbol{e}'_1, \boldsymbol{e}'_1\wedge\boldsymbol{e}'_2)$, and $(\boldsymbol{e}'_2\wedge\boldsymbol{e}'_3, \boldsymbol{e}'_3\wedge\boldsymbol{e}'_1, \boldsymbol{e}'_1\wedge\boldsymbol{e}'_2) = (\boldsymbol{e}_2\wedge\boldsymbol{e}_3, \boldsymbol{e}_3\wedge\boldsymbol{e}_1, \boldsymbol{e}_1\wedge\boldsymbol{e}_2)\tilde{T}$, where $\tilde{T}=(\det T)T$ is the cofactor matrix of T, which is an orthogonal matrix. If $\wedge^2(\boldsymbol{V}) \ni \boldsymbol{X}=(x_1, x_2, x_3)$ with respect to $(\boldsymbol{e}_2\wedge\boldsymbol{e}_3, \boldsymbol{e}_3\wedge\boldsymbol{e}_1, \boldsymbol{e}_1\wedge\boldsymbol{e}_2)$ and $\boldsymbol{X}=(x'_1, x'_2, x'_3)$ with respect to $(\boldsymbol{e}'_2\wedge\boldsymbol{e}'_3, \boldsymbol{e}'_3\wedge\boldsymbol{e}'_1, \boldsymbol{e}'_1\wedge\boldsymbol{e}'_2)$, then $(x'_1, x'_2, x'_3) = (x_1, x_2, x_3)(\det T)T$. Hence we define

Definition 1. A pseudo-vector is an element of $\wedge^2(\boldsymbol{V})$.

Now $\boldsymbol{e}_1\wedge\boldsymbol{e}_2\wedge\boldsymbol{e}_3$ and $\boldsymbol{e}'_1\wedge\boldsymbol{e}'_2\wedge\boldsymbol{e}'_3$ are bases of $\wedge^3(\boldsymbol{V})$, and $\boldsymbol{e}'_1\wedge\boldsymbol{e}'_2\wedge\boldsymbol{e}'_3 = (\det T)\,\boldsymbol{e}_1\wedge\boldsymbol{e}_2\wedge\boldsymbol{e}_3$. If an element ξ of $\wedge^3(\boldsymbol{V})$ is expressed as $\xi = x\boldsymbol{e}_1\wedge\boldsymbol{e}_2\wedge\boldsymbol{e}_3 = x'\boldsymbol{e}'_1\wedge\boldsymbol{e}'_2\wedge\boldsymbol{e}'_3$, then $x' = x(\det T)$.

Definition 2. A pseudo-scalar is an element of $\wedge^3(\boldsymbol{V})$.

Now, let U be a fixed differentiable manifold and W be a vector space (over $\boldsymbol{R}$). A C^1-map of U to W is called a *W-field*. The set $C^1(U, W)$ of all W-fields will be denoted by $F(W)$. The exterior differentiation d sends

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$F(\boldsymbol{R})$ to $F(V)$, $F(V)$ to $F(\wedge^2(V))$ and $F(\wedge^2(V))$ to $F(\wedge^3(V))$, thus it is in reality a common name of 3 maps $d_0: F(\boldsymbol{R}) \to F(V)$, $d_1: F(V) \to F(\wedge^2(V))$, $d_2: F(\wedge^2(V)) \to F(\wedge^3(V))$. Now the well-known operator $*$ interchanges $\boldsymbol{R}$ with $\wedge^3(V)$, V with $\wedge^2(V)$ with respect to an orientation of $\boldsymbol{E}$. It induces therefore 4 maps $*_0: F(\boldsymbol{R}) \to F(\wedge^3(V))$, $*_1: F(V) \to F(\wedge^2(V))$, $*_2: F(\wedge^2(V)) \to F(V)$, $*_3: F(\wedge^3(V)) \to F(\boldsymbol{R})$. From these we obtain 3 maps $\delta_p = (-1)^p *_{4-p} d_{3-p} *_p$, $p=1, 2, 3$, which will be simply denoted by δ.

Traditionally there are two well-known operators div and rot, each of which consists in reality of two operators: div of div_1 and div_2; div_1 is a map from $F(V)$ to $F(\boldsymbol{R})$ which coincides with our $-\delta_1 (=-\delta)$, and $\mathrm{div}_2 = d_2$: $F(\wedge^2(V)) \to F(\wedge^3(V))$; rot consists of $\mathrm{rot}_1 = d_1: F(V) \to F(\wedge^2(V))$ and $\mathrm{rot}_2 = \delta_2$: $F(\wedge^2(V)) \to F(V)$.

In this way, all the well-known operators d, $*$, div, rot are redefined and d, div, rot are free from frame and orientation.

Furthermore, we introduce the *co-exterior product* of a vector $\boldsymbol{a} \in V$ and a pseudo-vector $\boldsymbol{B} \in \wedge^2(V)$ by

$$\boldsymbol{B} \vee \boldsymbol{a} = *(*\boldsymbol{B} \wedge \boldsymbol{a}) \in V.$$

This is also frame-free.

Example (i) *Rotating frame and angular velocity.* Let $S: \boldsymbol{R} \to \boldsymbol{SO}$ be a C^2-map from $\boldsymbol{R}$ to the special orthogonal group $\boldsymbol{SO}$ of degree 3. If $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is a fixed orthonormal base of V, then $\boldsymbol{s}(t) = (\boldsymbol{s}_1(t), \boldsymbol{s}_2(t), \boldsymbol{s}_3(t)) = (\boldsymbol{e}_1, \boldsymbol{e}_2. \boldsymbol{e}_3)S(t)$ is another orthonormal base of V, and $(\boldsymbol{s}_2(t) \wedge \boldsymbol{s}_3(t), \boldsymbol{s}_3(t) \wedge \boldsymbol{s}_1(t), \boldsymbol{s}_1(t) \wedge \boldsymbol{s}_2(t))$ is a base of $\wedge^2(V)$. $A(t) = {}^tS(t)'S(t)$ is an antisymmetric matrix, belonging to the Lie algebra $\boldsymbol{so}(3)$. Let $\nu = \nu(t)$ be the isomorphism of $\wedge^2(V)$ to $\boldsymbol{so}(3)$ such that $\nu(\boldsymbol{s}_2(t) \wedge \boldsymbol{s}_3(t)) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$, $\nu(\boldsymbol{s}_3(t) \wedge \boldsymbol{s}_1(t)) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\nu(\boldsymbol{s}_1(t) \wedge \boldsymbol{s}_2(t)) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\Omega(t) = \nu^{-1}(A(t))$. This element of $\wedge^2(V)$ is called the angular velocity of the "rotating frame" $\Sigma(t) = (O; \boldsymbol{s}_1(t), \boldsymbol{s}_2(t), \boldsymbol{s}_3(t))$ rotating around O. Then we have $(d^2\boldsymbol{s}(t))/dt^2 = \Omega'(t) \vee \boldsymbol{s}(t) + \Omega(t) \vee (\Omega(t) \vee \boldsymbol{s}(t))$. This is the equation of motion expressed independently of orientation.

(ii) *Maxwell's equations.* $\boldsymbol{E}^T$ denotes the time-space, which is the naturally oriented 1-dimensional Euclidean space. $\boldsymbol{E} \times \boldsymbol{E}^T$ will be considered as U in §2. The so-called electric field, magnetic field, field of electric current density, and field of electric charge density, often denoted by $\boldsymbol{e}$, $\boldsymbol{B}$, $\boldsymbol{j}$ and ρ are in our language V-field, $\wedge^2(V)$-field, V-field and $\boldsymbol{R}$-field respectively. The electro-magnetic field is $(\boldsymbol{e}, \boldsymbol{B}, \boldsymbol{j}, \rho)$ and Maxwell's equations are written in the form

$$\text{(I)} \quad (\mathrm{rot}_1 \boldsymbol{e})(X, T) = -\frac{\partial \boldsymbol{B}(X, T)}{\partial T} \qquad \text{in } \wedge^2(V)$$

$$\text{(II)} \quad (\mathrm{rot}_2 \boldsymbol{B})(X, T) = \mu_0 \boldsymbol{j}(X, T) + \frac{1}{c^2} \frac{\partial \boldsymbol{e}(X, T)}{\partial T} \qquad \text{in } V$$

(III) $\varepsilon_0(\mathrm{div}_1\, \boldsymbol{e})(X, T)=\rho(X, T)$ in $\boldsymbol{R}$

(IV) $(\mathrm{div}_2\, \boldsymbol{B})(X, T)=0$ in $\wedge^3(\boldsymbol{V})$.

These equations are frame-free.

A part of this note was presented and distributed at the ICM 86, Berkeley. A detailed description is in [1].

References

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