32. *On Eisenstein Series of Degree Two*

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(Communicated by Shokichi IYANAGA, M. J. A., April 13, 1987)

Our motive is to see that the conditional theorem E in Introduction of [5] holds unconditionally for cusp forms. To do it, we study real analytic Eisenstein series with level. q denotes a fixed natural number $\geqq 3$ and we put $\Gamma_n = Sp_n(\boldsymbol{Z})$, $\Gamma_n(q)=\{M \in \Gamma_n \mid M \equiv 1_{2n} \bmod q\}$, $\Gamma_n(\infty)=\left\{\begin{pmatrix} * & * \\ 0^{(n)} & * \end{pmatrix} \in \Gamma_n\right\}$, and $H=\{Z \in M_2(\boldsymbol{C}) \mid Z={}^tZ,\ \mathrm{Im}\, Z>0\}$. If $C, D \in M_2(\boldsymbol{Z})$ satisfy $C{}^tD=D{}^tC$ and (C, D) is primitive, then we write $(C, D)=1$.

§1. **Definition of Eisenstein series and their relations.** For $Z \in H$ and $s \in \boldsymbol{C}$ we put

$$E(Z, s)=\sum_M |Y(M\langle Z\rangle)|^s,$$
$$E'(Z, s)=|Y|^s \sum_{C,D} (abs\,|CZ+D|)^{-2s}$$

where M runs over $(\Gamma_2(q) \cap \Gamma_2(\infty)) \backslash \Gamma_2(q)$, $Y(*)$ denotes the imaginary part of $*$, and (C, D) runs over $\Gamma_1(q) \backslash \{(C, D) \in M_{2,4}(\boldsymbol{Z}) \mid C{}^tD=D{}^tC,\ (C, D) \equiv (0, 1_2) \bmod q\}$. For $V=\begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}$, $V_i \in M_2(\boldsymbol{Q})$ such that qV, $qVI{}^tV\left(I=\begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}\right)$ are integral, we put

$$\tilde{E}'(Z, s\,; V)=|Y|^s \sum_{(C,D)} \exp(\pi i \operatorname{tr}((V_4, -V_3){}^t(2C-V_1, 2D-V_2)))$$
$$\cdot (abs\,|CZ+D|)^{-2s},$$

where (C, D) runs over $\Gamma_1(q) \backslash \{(C, D) \in M_{2,4}(\boldsymbol{Q}) \mid C{}^tD=D{}^tC,\ (C, D) \equiv (V_1, V_2) \bmod 1,\ rk(C, D)=2\}$.

Proposition 1. (i) $E'(Z, s)=(2\sharp SL_2(\boldsymbol{Z}/q))^{-1} \sum_a \{\sum_U (abs\,|U|)^{-2s}\} \times E(u(a)Z, s)$ *where* a, U *run over* $(\boldsymbol{Z}/q\boldsymbol{Z})^\times/\{\pm 1\}$, $\Gamma_1(q) \backslash \{U \in M_2(\boldsymbol{Z}) \mid a|U| \equiv \pm 1 \bmod q\}$ *respectively and* $u(a)$ *is an element of* Γ_2 *such that* $u(a) \equiv diag(1, a^{-1}, 1, a) \bmod q$.

(ii) $E(Z, s)=2\varphi(q)^{-1} \sum_a (\sum_\chi \chi(a) L(2s, \chi)^{-1} L(2s-1, \chi)^{-1}) E'(u(a)Z, s)$, *where* a *runs over the same set as above and* χ *runs over even Dirichlet characters modulo* q.

(iii) *For* $V=q^{-1}\begin{pmatrix} 0 & a1_2 \\ 0 & cJ \end{pmatrix} \in M_4(\boldsymbol{Q})$ *where* a, c *are relatively prime integers and* $J=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma_1$, $\tilde{E}'(Z, s\,; V)$ *is a linear combination of* $E(\sigma Z, s)(\sigma \in \Gamma_2)$. *If* $a=1$, $c=0$, *then* $\tilde{E}'(Z, s\,; V)=q^{4s}E'(Z, s)$.

§2. **Analytic continuation of Eisenstein series.** For $Z={}^tZ \in M_n(\boldsymbol{C})$ with a positive definite imaginary part, put

$$\sum (abs\,|Z+S|)^{-2s}=\sum_h \exp(2\pi i \operatorname{tr}(h\, \mathrm{Re}\, Z))\xi_n(\mathrm{Im}\, Z, h\,; s, s)$$

where S (resp. h) runs over integral (resp. half-integral) symmetric matrices of degree n. The function ξ_n is investigated in [11]. We need a sharper estimate than (4.13.K) in it:

$$|\omega_n(g, h; \alpha, \beta)| < Ae^{t^a}(1+\mu(h, g)^{-1})^{bt}e^{-\tau(h,g)}$$

if $|\alpha|, |\beta| < t$, where A, a, b are positive constants independent of g, h, α, β, and μ, τ are those in [11].

For $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_2$, $E(\sigma Z, s) = |Y|^s \sum (abs|CZ+D|)^{-2s}$, where (C, D) runs over $\Gamma_1(q)\backslash\{(C, D) \mid (C, D)=1,\ (C, D) \equiv (\gamma, \delta) \bmod q\}$. We denote by $E^{(i)}(\sigma Z, s)$ a partial sum with $rk\, C = i$.

Lemma 1. *Put $S = \{(C, D) \mid (C, D)=1,\ (C, D) \equiv (\gamma, \delta) \bmod q,\ rk\, C=1\}$. S is not empty if and only if there is a $v \in \Gamma_1$ such that*

$$\gamma^t v^{-1} \equiv \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \bmod q$$

for some $a, b \in \mathbf{Z}$, and then put $n = af - be$, where

$$\delta v = \begin{pmatrix} * & e \\ * & f \end{pmatrix}, \quad q_1 = (n, q) \quad and \quad q_2 = q/q_1.$$

For $w = (w_{ij}) \in \{w \in \Gamma_1 \mid w_{21} \equiv 0 \bmod q_2\}/\Gamma_1(\infty)$, $c, d \in \mathbf{Z}$ such that $(c, d)=1$, $c \equiv nw_{2,2}^2 \bmod q$, $d \equiv |\delta| \bmod q$, there is the only one $u \in \Gamma_1/\Gamma_1(q)$ such that

$$C := u\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}{}^t(vw) \equiv \gamma \bmod q, \qquad D := u\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}(vw)^{-1} \equiv \delta \bmod q,$$

and these C, D parametrized by w, c, d are complete representatives of $\Gamma_1(q)\backslash S$.

The Fourier expansion of $E^{(2)}(\sigma Z, s)$ is

$$q^{-4s}|\mathrm{Im}\, Z|^s \sum_h \exp(2\pi i\, \mathrm{tr}\,(h\, \mathrm{Re}\, Z)/q)\xi_2(q^{-1}\, \mathrm{Im}\, Z, h; s, s)\zeta(2s, h; \sigma),$$

where h runs over half-integral matrices of degree 2 and

$$\zeta(s, h; \sigma) = \sum_C (abs|C|)^{-s}\{\sum_D \exp(2\pi i\, \mathrm{tr}\,(hC^{-1}D)/q)\},$$

where C runs over $\Gamma_1(q)\backslash\{C \in M_2(\mathbf{Z}) \mid C \equiv \gamma \bmod q,\ |C| \neq 0\}$, D runs over

$\{D \in M_2(\mathbf{Z}) \bmod qC\Lambda \mid (C, D)=1,\ D \equiv \delta \bmod q\}(\Lambda = \{S = {}^tS \in M_2(\mathbf{Z})\})$.

Lemma 2. *Let $q = \prod_i p_i^{a_i}$ be a decomposition to primes and for a character χ modulo q, χ_i denotes a character modulo $p_i^{a_i}$ such that $\chi = \prod \chi_i$. Then we have*

$$\zeta(s, h; \sigma) = 2\varphi(q)^{-1}\sum_\chi \prod_{p \nmid q} b_p(\chi(p)p^{-s}, h) \prod_i b_{p_i}(\prod_{j \neq i} \chi_j(p_i)p_i^{-s}, h, \chi_i, \sigma),$$

where χ runs over even characters modulo q and

$$b_p(x, h) = \sum_C x^{\mathrm{ord}|C|} \sum_D \exp(2\pi i\, \mathrm{tr}\,(hC^{-1}D)),$$

where C runs over $\Gamma_1\backslash\{C \in M_2(\mathbf{Z}) \mid 0 < |C| \,|\, p^\infty\}$, D runs over $\{D \in M_2(\mathbf{Z}) \bmod C\Lambda \mid (C, D)=1\}$,

$$b_{p_i}(x, h, \chi_i, \sigma) = \sum_C x^{\mathrm{ord}|C|} \sum_D \exp(2\pi i\, \mathrm{tr}\,(hC^{-1}D)p_i^{-a_i}) \sum_R \overline{\chi_i(|R|)},$$

where C, D, R run over $\Gamma_1\backslash\{C \in M_2(\mathbf{Z}) \mid 0 < |C| \,|\, p_i^\infty\}$, $\{D \in M_2(\mathbf{Z}) \bmod p_i^{a_i}C\Lambda \mid (C, D)=1\}$ and $\{R \in GL_2(\mathbf{Z}/p_i^{a_i}\mathbf{Z}) \mid R(C, qp_i^{-a_i}D) \equiv (\gamma, \delta) \bmod p_i^{a_i}\}$ respectively. $b_{p_i}(x, h, \chi_i, \sigma)$ is a rational function in x and the denominator is $(1-p_i^3x^2)(1-p_i^2x)$ if $\chi_i = 1$, $h=0$; $(1-p_i^3x^2)$ if either $\chi_i \neq 1$, $\chi_i^2 = 1$, $h=0$ or $\chi_i^2=1$, $rk\, h=1$; 1 other-

wise, and the degree and coefficients of the numerator is $O(p_i^{AH})$ where A is a positive constant independent of h and H is a sum of p_i-order of non-zero elementary divisors of h.

Using these lemmas, we can express $E(\sigma z, s)$ by L-functions, Γ-functions and $\xi_i(*, h; *, *)$ $(h \neq 0)$ and we have

Theorem. *Let $\sigma \in \Gamma_2$ and $Z \in H$ be in the Siegel's fundamental domain. Then $(s-1/2)(s-3/4)(s-1)^2(s-3/2) \prod_\chi L(2s, \chi) \cdot \prod_{\chi'} L(4s-2, \chi')E(\sigma Z, s)$ is an entire function, and if $|s|<t$, then it is $O(e^{t^a}|\mathrm{Im}\, Z|^{t^a})$ for some $a>0$ independent of t. Here χ, χ' run over even characters modulo q, squares of even characters modulo q respectively.*

§3. **Functional equations.** Put $H^* = \{W \in M_2(\mathbf{R}) \mid W = T+S,\ T = {}^tT > 0,\ S = -{}^tS\}$,

$$\Delta^*(q) = \{M \in SL_4(\mathbf{Z}) \mid M \equiv 1_4 \bmod q,\ {}^tMI^*M = I^*\} \left(I^* = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}\right).$$

For $Z \in H$, $W \in H^*$, $V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} (V_i \in M_2(\mathbf{R}))$, we put

$$\theta(Z, W; V) = \sum_{C, D \in M_2(\mathbf{Z})} \exp(-\pi \operatorname{tr}((\mathrm{Im}\, Z)^{-1}\{Z^t(C+V_1) + {}^t(D+V_2)\}W) + \pi i \operatorname{tr}((V_4, -V_3)^t(2C+V_1, 2D+V_2))).$$

Suppose that qV, qVI^tV are integral; then

$$\begin{aligned} \xi(Z, s; V) &:= \int_{\Delta^*_\infty(q)\backslash H^*} |T|^{s+1/2} R^*\theta(Z, W; V)\, dW \\ &= q\pi^{1/2-2s}(s-3/2)(s-1)\Gamma(s+1)\Gamma(s+1/2)\tilde{E}'(Z, s; V) \\ &= \int_{\Delta^*(q)\backslash H^*} G^*(W, s-1/2)|T| R^*\theta(Z, W; V)\, dW, \end{aligned}$$

where $\Delta^*_\infty(q) = \left\{\begin{pmatrix} * & * \\ 0^{(2)} & * \end{pmatrix} \in \Delta^*(q)\right\}$, $T = 2^{-1}(W + {}^tW)$, and R^*, dW are a differential operator and an invariant volume element given in [8] respectively. For $N = \begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix} \in \Delta^*(1)$, there are $U \in GL_2(\mathbf{Z})$, $a, b \in \mathbf{Z}$ such that

$$(\gamma, \delta) \equiv \left(aU\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, bU\right) \bmod q,\ (a, b) = 1,$$

and then

$$G^*(N\langle W \rangle, s) = \sum_k E_1(w, 2s; (ka, kb))$$

where k runs over $\{k \bmod q \mid k^2 \equiv \pm 1 \bmod q\}/\{\pm 1\}$ and $w = u+iv$, $(2^{-1}(W - {}^tW) = \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix}$, $v = \sqrt{|2^{-1}(W + {}^tW)|}$), $E_1(w, s; (a, b)) = v^s \sum |cw+d|^{-2s}$ where c, d run over $\{(c, d) \mid (c, d) = 1,\ c \equiv a \bmod q,\ d \equiv b \bmod q\}$. Functional equations for E_1 ([6]) imply ones for G^* and then for $E(Z, s)$.

Theorem. *$E(Z, s) = \sum_\sigma \varphi(s, \sigma)E(\sigma Z, 3/2-s)$, where σ runs over left cosets $\langle \Gamma_2(q), \Gamma_2(\infty) \rangle \backslash \Gamma_2$ of Γ_2 by a subgroup generated by $\Gamma_2(q)$ and $\Gamma_2(\infty)$. Here $\varphi(s, \sigma) = 2^{5-4s}q^{-3}\pi^{5/2}\Gamma(s)^{-2}\Gamma(s-1/2)^{-2}\Gamma(2s-3/2)\Gamma(2s-2)\zeta(2s, 0; \sigma^{-1})$.*

Remark. $\zeta(2s, 0; \sigma) = \sum_\chi L(2s-2, \tilde{\chi})L(4s-3, \widetilde{\chi^2})L(2s, \chi)^{-1}L(4s-2, \chi^2)^{-1} \times F(\chi, \sigma)$, where χ runs over even characters modulo q and $\tilde{*}$ is a primitive character induced by $*$ and $F(\chi, \sigma)$ is a polynomial in $p^{-s}(p\,|\,q)$ dependent on χ and σ.

Corollary. *Let* $f(Z)=\sum a(N)\exp(2\pi i\,\mathrm{tr}\,(NZ)/q)$ *be a cusp form of level* q, *weight* k *of degree* 2. *Then we have* $a(N)=O(|N|^{k/2-1.5/(a+5b+10)})$ *where* a, b *are the number of even characters modulo* q, *of squares of even characters modulo* q *respectively.*

Remark 1. The estimate in Corollary will be improved, transforming $f(Z)$ to Γ_0-type cusp forms.

Remark 2. Using [10], we can get functional equations for more general Eisenstein series of degree 2.

Remark 3. An analytic continuation for $E(-Z^{-1}, s)$ is easier and the method here works for Eisenstein series of higher degree, using [4] instead of [3].

Details will appear elsewhere.

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