31. Degeneration of Kunev Surfaces. I

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0. The purpose of this note is to outline our recent results on degeneration of Kunev surfaces. Details will be published elsewhere.

A Kunev surface is, by definition (see 1 below), a double cover of a K3 surface. We report descriptions of degenerations of Kunev surfaces over some fixed K3 surfaces (Theorems 1 and 2). These theorems have an interesting application: We can explain in a uniform way the failure of the Torelli theorem for Kunev surfaces and elliptic surfaces with $p_g=1$ and $q=0, 1$ (Corollary 3). We use the terminology a *homotopic K3 surface* and an *elliptic surface* as ones with $\kappa=1$.

1. A *Kunev surface* is defined as a minimal surface X of general type with $p_g=c_1^2=1$ which has an involution σ such that $Y':=X/\sigma$ is a K3 surface with rational double points (R.D.P. for short) and the bicanonical map of X is a Galois cover of $\boldsymbol{P}^2$ factoring through Y'. Let X be a Kunev surface with ample K_X. Then it is known that the branch locus $B\subset\boldsymbol{P}^2$ of the bicanonical map consists of two smooth cubics C_j $(j=1, 2)$ and of a line L such that $B=\sum C_j+L$ has only nodes as singularities (see [1], [6]), and X is reconstructed as follows: (i) Take the double cover Y' of $\boldsymbol{P}^2$ branched over $\sum C_j$. (ii) Take the minimal resolution Y of Y'. (iii) Take the double cover $\tilde{X}$ of Y branched over $L+\sum E_j$, where E_j $(1\leq i\leq 9)$ are (-2)-curves appeared in (ii). (iv) Contracting (-1)-curves on $\tilde{X}$ induced from E_i, we recover the Kunev surface X.

2. Horikawa and Shah constructed a completion of the moduli space of K3 surfaces of degree 2 as a completion of {sextics in $\boldsymbol{P}^2$} by geometric invariant theory ([3], [5]), which contains our K3 surfaces Y appeared in 1. The latter form 10-dimensional submoduli $\mathfrak{N}$ over which sits "a completion" of the moduli space $\mathfrak{M}$ of Kunev surface . The first theorem is concerned with a completion of the fiber over a general point in $\mathfrak{N}$. Let C_1 and C_2 be general cubics in $\boldsymbol{P}^2$. Denote by $\check{C}_j\subset\check{\boldsymbol{P}}^2$ the dual curve of C_j $\subset\boldsymbol{P}^2$, i.e., the image of the Gauss map. Then each $\check{C}_j$ has nine cusps corresponding to nine inflexes on C_j, $\sum\check{C}_j$ has nine bitangents $\check{D}_i$ with tangent points P_{i1} and P_{i2} $(1\leq i\leq 9)$ subjected to nine nodes of $\sum C_j$, and we have two stratifications of $\check{\boldsymbol{P}}^2$ determined by $\sum\check{C}_j$ and $\sum\check{D}_i$:

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$$\check{P}^2=(\check{P}^2-\sum \check{C}_j)\cup(\sum \check{C}_j-(\sum \check{P}_{ji}+\mathrm{Sing}(\sum \check{C}_j))\cup(\sum \mathrm{Sing}(\check{C}_j))$$
$$\cup(\cap \check{C}_j)\cup(\sum \check{P}_{ji})$$
$$=:R_0\cup R_1\cup R_1'\cup R_2\cup R_0'.$$
$$\check{P}^2=(\check{P}^2-\sum \check{D}_i)\cup(\sum \check{D}_i-\mathrm{Sing}(\sum \check{D}_i))\cup \mathrm{Sing}(\sum \check{D}_i)$$
$$=:S_0\cup S_1\cup S_2.$$

Theorem 1. *With the above notation, there exists a complete family* $f:\mathcal{X}\to\check{P}^2$ *of degenerations of Kunev surfaces over the fixed general point* $[\sum C_j]\in\mathfrak{N}$. *This family has the following properties:*

(1.1) *The singularity of the total space* $\mathcal{X}$ *consists of mutually disjoint compounds Veronese cone over* $\check{D}_i$ $(1\le i\le 9)$, *i.e., analytically isomorphic to the product of* $\check{D}_i$ *and the cone over the Veronese embedding of* $P^2\subset P^5$ *by* $|\mathcal{O}_{P^2}(2)|$. *Hence a single blowing-up along the singular loci yields a resolution* $\tilde{f}:\tilde{\mathcal{X}}\to\check{P}^2$. *For each* i $(1\le i\le 9)$, *the exceptional divisor* $\mathcal{W}_i$ *is a family of* P^2 *over* $\check{D}_i$. *The universal family* $\{L_t\,|\,t\in\check{P}^2\}$ *of lines on* P^2 *induces an irreducible divisor on* $\tilde{\mathcal{X}}$. *We denote by* $\mathcal{L}$ *the divisor endowed with reduced structure. Then* $K_{\tilde{\mathcal{X}}}=\mathcal{L}+\sum \mathcal{W}_i$.

(1.2) *Besides the singularity of the total space* $\mathcal{X}$, *the fiber* X_t *has R.D.P. raised from the tangent points of* L_t *and* $\sum C_j$ *on* P^2. *These singularities form two disjoint compounds* A_1 *over* $R_1\cup R_2$, *which degenerate to* A_2 *over* R_1'. *Over* R_0', *clash an* A_1 *and a Veronese cone singularity. The effect is explained in* (1.6) *below.*

(1.3) *The fiber* $\tilde{X}_t=V_t+\sum W_{i,t}$, *where* V_t *is the main component and the summation runs over the indices i for which* $t\in\check{D}_i$. *Hence the canonical curve* K_t *of* V_t *coincides with* $\mathcal{L}|V_t$.

(1.4) V_t *is a (singular) Kunev surface, homotopic K3 surface, or K3 surface according to* $t\in S_0, S_1$, *or* S_2.

(1.5) K_t *is irreducible reduced and passes* Sing(V_t), *if exists, and its geometric genus is* $2-(m+n)$ *for* $t\in(R_m\cup R_m')\cap S_n$.

(1.6) *In case* $t\in S_1-R_0'$, $V_t\cap W_{i,t}$ *is a smooth conic on* $W_{i,t}\simeq P^2$ *and a rational curve with selfintersection* -4 *on* V_t, *where* $t\in\check{D}_i$. *Whereas, in case* $t\in R_0'$, $V_t\cap W_{i,t}$ *decomposes into two distinct lines on* $W_{i,t}$ *and two rational curves with selfintersection* -3 *on* V_t.

Remark. (1) Since the isotropy group Isot$[\sum C_j]$ of $[\sum C_j]$ in PGL_2 is trivial, $\check{P}^2$ is actually a completion of the fiber of $\mathfrak{M}\to\mathfrak{N}$ over $[\sum C_j]$. (2) We can compute easily the following numbers: $\sharp(R_1)=9\cdot 2=18$, $\sharp(R_2)=6^2=36$, $\sharp(R_0')=9\cdot 2=18$, $\sharp(R_1\cap S_1)=4\cdot 9=36$, $\sharp(S_2)=9\cdot 8/2=36$. (3) We can describe easily a semi-stable reduction of a family induced over a disc.

3. Among the special cases with finite local monodromy in the pure second cohomology, we report here one of the most interesting cases. Let C_1 (resp. C_2) consists of three distinct lines $\sum M_k$ (resp. $\sum N_l$) passing through a common point T_1 (resp. T_2) such that $C_1\cap C_2$ are nine nodes D_i $(1\le i\le 9)$. Denote by $\check{M}_k$ and $\check{N}_l$ (resp. $\check{T}_j$ and $\check{D}_i$) the dual points (resp. lines) on $\check{P}^2$. Then the three points $\check{M}_k$ (resp. $\check{N}_l$) are on the line $\check{T}_1$ (resp. $\check{T}_2$), $\check{D}_i$ are the lines joining the points $\check{M}_k$ and $\check{N}_l$, and these determine a stratification on $\check{P}^2$;

$$\begin{aligned}\check{P}^2 &= (\check{P}^2-(\sum \check{D}_i+\sum \check{T}_j)) \cup (\sum \check{D}_i - \mathrm{Sing}\,(\sum \check{D}_i)) \\ &\cup (\mathrm{Sing}\,(\sum \check{D}_i)-(\sum \check{M}_k+\sum \check{N}_l)) \\ &\cup (\sum \check{T}_j-(\check{T}_1\cap \check{T}_2+\sum \check{M}_k+\sum \check{N}_l)) \cup (\check{T}_1\cap\check{T}_2) \cup (\sum \check{M}_k+\sum \check{N}_l) \\ &=: S_0\cup S_1\cup S_2\cup S_1'\cup S_2'\cup S_2''.\end{aligned}$$

Theorem 2. *With the above notation, there exists a complete family* $f:\mathfrak{X}\to\check{P}^2$ *of degenerations of Kunev surfaces over the fixed* $[\sum C_j]\in\mathfrak{N}$ *as above. This family has the following properties:*

(2.1) *Let* $\tilde{\mathfrak{X}}$ *be the blowing-up of* $\mathfrak{X}$ *along nine disjoint compounds Veronese cone over* $\check{D}_i$ $(1\le i\le 9)$ *raised from* $C_1\cap C_2$. *Then the same statement as* (1.1) *holds, provided that* $\tilde{\mathfrak{X}}$ *still has singularity described in* (2.2) *below.*

(2.2) *Rising from two triple points* T_1 *and* T_2, $\mathfrak{X}$ *has four compounds R.D.P. of type* D_4 *over* $\check{P}^2-\sum \check{T}_j=S_0\cup S_1\cup S_2$, *each two of which clash to make up a compound elliptic singularity on* $f^{-1}(\check{T}_j-S_2'')$ $(j=1,2)$ *with a local equation*

$$z^2+y(x^4+y^2)=0.$$

In case $t\in S_2''$, *say* $t=\check{M}_1$, *besides the two* D_4 *raised from* T_2, *the main component* V_t *of the fiber* $\tilde{X}_t$ *has the following singularity: We abuse the notation* T_1 *for the point on* V_t *induced from* $T_1\in P^2$. V_t *has ordinary double points along* $\mathcal{L}|V_t-T_1$ *and a local equation at* $T_1\in V_t$ *is*

$$z^2+y^2(x^2+y^4)=0.$$

Hence T_1 *becomes an R.D.P. of type* A_3 *on the normalization of* V_t.

(2.3) *The same statement as* (1.3) *holds.*

(2.4) *Analogously as* (1.4), V_t *is a singular Kunev surface, homotopic K3 surface, or K3 surface according to* $t\in S_0, S_1$, *or* S_2. *Whereas* V_t *becomes a singular elliptic surface with* $p_g=q=1$, *abelian surface, or K3 surface according to* $t\in S_1', S_2'$, *or* S_2''.

(2.5) *The canonical curve* K_t *on* V_t *is divided into two disjoint* (-1)-*curves in the case that* $t\in S_2$ *and that* t *is a triple point of* $\sum \check{D}_i$. K_t *becomes a double rational curve in case* $t\in S_2''$. *In other cases,* K_t *is irreducible reduced, and its geometric genus is* $2-n$ *for* $t\in S_n\cup S_n'$. K_t *passes the elliptic singular point or its degenerating point in case* $t\in\sum \check{T}_j=S_1'\cup S_2'\cup S_2''$.

(2.6) *An analogous statement as* (1.6) *holds according to* $t\in S_1-S_2''$, *or* S_2''. *In case* $t\in S_2''$, $V_t\cap W_{i,t}$ *on* V_t *consists of two rational curves which cut the double curve* $\mathcal{L}|V_t$ *transversely at a common point.*

Remark. We can give parallel remarks as those just after Theorem 1. We omit all but the version of (1).

(1′) $\mathrm{Isot}\,[\sum C_j]$ is a finite group and $\check{P}^2/\mathrm{Isot}\,[\sum C_j]$ is a completion of the fiber of $\mathfrak{M}\to\mathfrak{N}$ over $[\sum C_j]$.

The proofs of Theorems 1 and 2 go on the same way as the construction of smooth Kunev surfaces with ample K over P^2 explained in **1**. In order to prove (1.4) and (2.4), we use the elliptic fibration on the minimal model of V_t, for $t\in\check{D}_i$ or $\check{T}_j$, induced from the pencil of lines $\{L_s\,|\,s\in\check{D}_i\}$ or

$\{L_s \mid s \in \check{T}_j\}$ on $\boldsymbol{P}^2$.

4. Combining the Clemens-Schumid exact sequence (see [2]), we can explain uniformly the failure of Torelli theorem for the period map Φ_2 of the pure second cohomology of Kunev surfaces and elliptic surfaces with $p_g=1$ and $q=0, 1$ (cf. [7], [8], [9], [4]).

Corollary 3. *S_0, S_1 and S_1' in Theorems* 1 *and* 2 *appear as the fibers of the period map Φ_2 for Kunev surfaces, homotopic K*3 *surfaces and elliptic surfaces with $p_g=q=1$ respectively.*

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