29. On the Globally Asymptotic Stability of Solutions of Ordinary Differential Equations

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1. Introduction. We investigate the globally asymptotic stability of the zero solution of the ordinary differential equation

$$(1) \qquad \dot{x}=X(t,x) \qquad (X(t,0)\equiv 0),$$

where $X:\boldsymbol{R}^+\times\boldsymbol{R}^n\to\boldsymbol{R}^n$ is a continuous function and $\boldsymbol{R}^+=[0,+\infty)$.

In the autonomous case, that is, $X(t,x)\equiv X(x)$ in (1), Barbashin and Krasovski established conditions for uniformly asymptotic stability of the zero solution of (1) (see [4]). Some generalizations of their result to the nonautonomous differential equation (1) were given by Matrosov [3], Hatvani [1], [2], and [4], etc.

In this paper we extend Hatvani's results [2] and obtain the sufficient conditions for the globally asymptotic stability, globally equi-asymptotic stability, and globally uniformly asymptotic stability as well as uniform stability of the zero solution of (1).

2. Theorems. For $x\in\boldsymbol{R}^n$ and $\varepsilon>0$, let $B_n(x,\varepsilon)=\{y\in\boldsymbol{R}^n:\ \|y-x\|<\varepsilon\}$. The ε-neighborhood of a set $E\subset\boldsymbol{R}^n$ is the set $B_n(E,\varepsilon)=\{x\in\boldsymbol{R}^n: d(x,E)<\varepsilon\}$, where $d(x,E)=\inf\{\|x-y\|: y\in E\}$ is the distance from $x\in\boldsymbol{R}^n$ to E.

A function a is said to belong to class K ($a\in K$) if a is a continuous, strictly increasing function on $\boldsymbol{R}^+$ into $\boldsymbol{R}^+$ with $a(0)=0$.

A measurable function $\phi:\boldsymbol{R}^+\to\boldsymbol{R}^+$ is said to be integrally positive (see [1], [2], [3]) if

$$\int_I \phi(t)dt=+\infty$$

on every set $I=\bigcup_{i=1}^{+\infty}[\alpha_i,\beta_i]$ such that $\alpha_i<\beta_i<\alpha_{i+1}$, $\beta_i-\alpha_i\geqq\mu>0$ for $i=1,2,\cdots$. If, in addition, $\alpha_{i+1}-\beta_i\leqq\lambda$ $(i=1,2,\cdots)$ for some constant $\lambda>0$, ϕ is said to be weakly integrally positive.

Let a continuous function $Q:\boldsymbol{R}^+\times\boldsymbol{R}^n\to\boldsymbol{R}^m$ satisfy a locally Lipschitz condition in x. The derivative of Q with respect to the equation (1) is the function defined by

$$\dot{Q}_{(1)}(t,x)=\limsup_{h\to 0+}\frac{1}{h}[Q(t+h,\ x+hX(t,x))-Q(t,X)]$$
$$((t,x)\in\boldsymbol{R}^+\times\boldsymbol{R}^n).$$

For $p\in\boldsymbol{R}$, $[p]_+=\max\{p,0\}$ is said to be the positive part of p.

Let $x(\cdot\,;t_0,x_0)$ be a solution of (1) passing through a point (t_0,x_0) in $\boldsymbol{R}^+\times\boldsymbol{R}^n$.

Theorem 1. *Suppose that there exist an absolutely continuous func-*

tion $A: \boldsymbol{R}^+\times \boldsymbol{R}^n \to \boldsymbol{R}$ *and continuous functions* $V: \boldsymbol{R}^+\times \boldsymbol{R}^n \to \boldsymbol{R}$, $W: \boldsymbol{R}^n \to \boldsymbol{R}^m$, *which are locally Lipschitzian in* x, *such that for some* $a, b \in K$, *the following conditions hold.*

(I) $a(\|x\|) \leqq V(t,x) \leqq b(\|x\|)$ *in* $\boldsymbol{R}^+\times \boldsymbol{R}^n$, $a(r)\to +\infty$ $(r\to+\infty)$.

Let H *be any positive constant.*

(II) *There exist a integrally positive function* $\phi: \boldsymbol{R}^+\to \boldsymbol{R}^+$ *and a continuous function* $U: B_n(0,H)\to \boldsymbol{R}^+$ *such that*

$$\dot{V}_{(1)}(t,x) \leqq -\phi(t)U(x) \qquad \text{in } \boldsymbol{R}^+\times B_n(0,H).$$

(III) *Let* $F=U^{-1}(0)$. *For every compact set* $M\subset B_n(0,H)\backslash F$, *there exists a constant* $\rho=\rho(M)>0$ *such that* $\overline{B_n^*(M,\rho)}\cap F=\emptyset$, *where* $B_n^*(M,\rho) = W^{-1}[B_m(W(M),\rho)]\cap B_n(0,H)$.

(IV) *For every continuous function* $u: \boldsymbol{R}^+ \longrightarrow B_n^*(M,\rho)$,

$$\int_0^t \dot{W}_{(1)}(u(s))ds$$

is uniformly continuous in $\boldsymbol{R}^+$.

(V) *For any* $t_0 \in \boldsymbol{R}^+$ *and any* α_1, α_2 $(0<\alpha_1<\alpha_2<H)$, *there exist positive constants* β, c_1 *and a continuous function*

$$\psi: \boldsymbol{R}^+ \longrightarrow \boldsymbol{R} \qquad \left(\int_0^{+\infty} \psi(t)dt=+\infty\right)$$

such that for every continuous function $v: \boldsymbol{R}^+\to J_n(\alpha_1,\alpha_2)$,

$$A(t,v(t))\leqq c_1 \quad \text{and} \quad \dot{A}_{(1)}(t,v(t))\geqq \psi(t) \qquad \text{in } [t_0,+\infty),$$

where $J_n(\alpha_1,\alpha_2)=\{x\in \overline{B_n(F,\beta)}: \alpha_1\leqq \|x\|\leqq \alpha_2\}$.

Then the zero solution of (1) *is uniformly stable and globally attractive, therefore it is globally asymptotically stable.*

Theorem 2. *If, in addition to the assumptions in Theorem* 1, *every solution of* (1) *starting from a point in* $\boldsymbol{R}^+\times B_n(0,H)$ *is unique to the right, then the zero solution of* (1) *is uniformly stable and globally equi-asymptotically stable, therefore it is globally equi-asymptotically stable.*

Corollary 1. *Suppose that the function* ϕ *in* (II) *satisfies*

$$\int_0^{+\infty} \phi(t)dt=+\infty$$

instead of the integrally positive property. Furthermore, let (IV) *be replaced by the following.*

(IV′) *For every continuous function* $u: \boldsymbol{R}^+\to B_n^*(M,\rho)$,

$$\int_0^{+\infty} \|\dot{W}_{(1)}(u(s))\|ds<+\infty.$$

Then the statements of Theorems 1 *and* 2 *remain true.*

Corollary 2. *Suppose that the function* ϕ *in* (II) *is only weakly integrally positive and* (V) *is replaced by the following.*

(V′) *For any constants* α_1 *and* α_2 $(0<\alpha_1<\alpha_2<H)$, *there exist positive constants* β, c_2 *and* c_3 *such that*

$$|A(t,x)|\leqq c_2 \quad \text{and} \quad \dot{A}_{(1)}(t,x)\geqq c_3 \qquad \text{in } \boldsymbol{R}^+\times J_n(\alpha_1,\alpha_2).$$

Then the statements of Theorems 1 *and* 2 *remain true.*

Theorem 3. *Suppose that all the assumptions in Theorem* 1 *except for* (II) *and* (V) *hold. Let* (II) *be replaced by the following.*

(II′) *There exist a positive constant* c_4 *and a continuous function* $U: B_n(0, H) \to \boldsymbol{R}^+$ *such that*

$$\dot{V}_{(1)}(t, x) \leqq -c_4 U(x) \qquad \text{in } \boldsymbol{R}^+ \times B_n(0, H).$$

If, in addition, (V′) *is satisfied, then the zero solution of* (1) *is globally uniformly asymptotically stable.*

Corollary 3. *In the above theorems and corollaries, let* $\dot{W}_{(1)}(u(s))$ *in assumptions* (IV) *and* (IV′) *be replaced by the function* $[\dot{W}_{(1)}(u(s))]_+$, *then the statements of Theorems* 1–3 *and Corollaries* 1–2 *remain true.*

3. Proofs. To prove Theorems and Corollaries, we need the following lemmas obtained by Hatvani [1].

Lemma 1. *Let* H *be some positive constant. Suppose that there exist continuous functions* $V: \boldsymbol{R}^+ \times B_n(0, H) \to \boldsymbol{R}$ *and* $W: B_n(0, H) \to \boldsymbol{R}^m$, *which are locally Lipschitzian in* x. *Let* F *be a subset of* $B_n(0, H)$ *and* $0 < H' < H$. *For any* $r \in B_n(0, H) \setminus F$, *there exist* $\rho = \rho(r) > 0$ *and* $T = T(r) \in \boldsymbol{R}^+$ *such that for any continuous function* $u: [T, +\infty) \to B_n^*(r, \rho) \cap B_n(0, H')$, $(B_n^*(r, \rho) = W^{-1}[B_m(W(r), \rho)])$, *the following conditions hold.*

(i) $\int_T^t \dot{W}_{(1)}(u(s))ds$ *is uniformly continuous in* $[T, +\infty)$.

(ii) $\dot{V}_{(1)}(t, u(t))$ *is integrally negative in* $[T, +\infty)$.

(iii) $V(t, u(t))$ *is bounded from below in* $[T, +\infty)$.

Then for a solution $x(\cdot)$ *of* (1) *such that* $x(t) \in B_n(0, H')$ *in the right maximal interval* $[t_0, \omega)$ $(\subset \boldsymbol{R}^+)$ *where* $x(\cdot)$ *is defined, the positive limit set* Ω^+ *of* $x(\cdot)$ *is included in the set* F (i.e. $\Omega^+ \subset F$).

Lemma 2. *Suppose that conditions* (i) *and* (ii) *in Lemma* 1 *are replaced by the following* (i′) *and* (ii′), *respectively.*

(i′) $\int_T^{+\infty} \| \dot{W}_{(1)}(u(s)) \| ds < +\infty$

(ii′) $\int_T^{+\infty} \dot{V}_{(1)}(t, u(t))dt = -\infty$ *and* $\dot{V}_{(1)}(t, u(t)) \leqq 0$ *in* $[T, +\infty)$.

Then the statement of Lemma 1 *remains true.*

Proof of Theorem 1. Conditions (I) and (II) imply that the zero solution of (1) is uniformly stable and all solutions of (1) are uniformly bounded. Therefore, for any $x_0 \in \boldsymbol{R}^n$, there exists $H' > 0$ such that for every $t_0 \in \boldsymbol{R}^+$ and every solution $x(\cdot\,; t_0, x_0)$,

$$(2) \qquad \|x(t; t_0, x_0)\| < H' \qquad \text{for } t \in [t_0, +\infty).$$

From the uniform stability, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for every $t_1 \in \boldsymbol{R}^+$, $x_1 \in B_n(0, \delta)$, and any solution $x(\cdot\,; t_1, x_1)$ of (1), $\|x(t; t_1, x_1)\| < \varepsilon$ for $t \geqq t_1$.

Let $\alpha_1 = \delta$ and $\alpha_2 = H'$. Choose $H > 0$ such that $H' < H$. Now all the assumptions in Lemma 1 are satisfied. Hence the positive limit set Ω^+ of $x(\cdot\,; t_0, x_0)$ belongs to F. Thus there exists $\tau \geqq t_0$ such that

$$(3) \qquad x(t; t_0, x_0) \in B_n(F, \beta) \qquad \text{for } t \in [\tau, +\infty).$$

Now we show that there exists $T = T\,(t_0, \varepsilon, x_0, x(\cdot\,; t_0, x_0)) > 0$ such that $\|x(t_0 + T; t_0, x_0)\| < \delta$. If it is not true, then by (2),

(4) $$\delta \leqq \|x(t; t_0, x_0)\| < H' \quad \text{in } [t_0, +\infty).$$

(3) and (4) imply that $x(t; t_0, x_0) \in J_n(\alpha_1, \alpha_2)$ for $t \in [\tau, +\infty)$. Hence by (V), for any $t \geqq \tau$,

(5) $$c_1 \geqq A(t, x(t)) = A(\tau, x(\tau)) + \int_\tau^t \dot{A}_{(1)}(s, x(s))ds$$
$$\geqq A(\tau, x(\tau)) + \int_\tau^t \psi(s)ds,$$

where $x(\cdot) = x(\cdot\,; t_0, x_0)$. This contradicts the fact that

$$\int_\tau^{+\infty} \psi(s)ds = +\infty.$$

Thus, $\|x(t; t_0, x_0)\| < \varepsilon$ for $t \geqq t_0 + T$. Therefore the origin is globally asymptotically stable. Q.E.D.

Proof of Theorem 2. From Theorem 1, the zero solution is globally asymptotically stable. Thus, by the uniqueness assumption, for any $t_0 \geqq 0$, $\eta > 0$, $\varepsilon > 0$, and every $x_0 \in B_n(0, \eta)$, there exists $T = T(t_0, \eta, \varepsilon, x_0,) > 0$ such that $\|x(t_0 + T; t_0, x_0)\| < \delta = \delta(\varepsilon)$, where δ is defined in the proof of Theorem 1. It also follows from the uniqueness assumption that the solution $x(\cdot\,; t_0, x_0)$ is continuous in x_0. From this and the fact that $\overline{B_n(0, \eta)}$ is compact, T can be chosen as the one independent of x_0. Therefore the zero solution is globally equi-asymptotically stable. Q.E.D.

If Lemma 2 instead of Lemma 1 is used in the proofs of Theorems 1 and 2, then we can prove Corollary 1.

To prove Corollary 2 and Theorem 3, we use the compact set $M = \{x \in \boldsymbol{R}^n : \delta \leqq \|x\| \leqq H',\ x \notin B_n(F, \beta)\}$, where δ, H' are defined in the proof of Theorem 1 and β is given in (V′) for $\alpha_1 = \delta$ and $\alpha_2 = H'$.

The detailed proof will be published later.

References

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