27. *Path Integral for the Weyl Quantized Relativistic Hamiltonian*[†)]

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1. Introduction. The aim of this note is to give a path integral representation of the solution of the Cauchy problem for

(1.1) $$\partial_t \psi(t, x) = -[H - mc^2]\psi(t, x), \quad t>0, \quad x \in \boldsymbol{R}^d.$$

Here c is the light velocity. H is the quantum Hamiltonian via the Weyl correspondence, i.e. the pseudo-differential operator ([1], [2], [6])

(1.2) $$(Hg)(x) = (2\pi)^{-d} \iint_{\boldsymbol{R}^d \times \boldsymbol{R}^d} e^{i(x-y)p} h\left(p, \frac{x+y}{2}\right) g(y)\,dy\,dp, \qquad g \in \mathcal{S}(\boldsymbol{R}^d),$$

associated with the classical Hamiltonian

(1.3) $$h(p, x) = [(cp - eA(x))^2 + m^2c^4]^{1/2} + e\Phi(x), \quad p \in \boldsymbol{R}^d, \quad x \in \boldsymbol{R}^d,$$

of a relativistic spinless particle of mass $m>0$ and charge e interacting with electromagnetic vector and scalar potentials $A(x)$ and $\Phi(x)$ (e.g. [5]). The Planck constant $\hbar$ is taken to equal 1.

The present approach is a rigorous application of the phase space path integral or Hamiltonian path integral method with the "midpoint" prescription ([6], [7]). The path space measure used is a probability measure on the space of the right-continuous paths $X : [0, \infty) \to \boldsymbol{R}^d$ having the left-hand limits. Each path $X(s)$ is called a d-dimensional time homogeneous Lévy process ([3], [4]). The path integral formula obtained has a close analogy with the Feynman-Kac-Itô formula for the quantum Hamiltonian of a nonrelativistic spinless particle of the same mass and charge interacting with vector and scalar potentials (e.g. [8]).

2. Path integral representation. To formulate our result we need some notions from a time homogeneous Lévy process ([3], [4]). The path space measure which we are going to use is the probability measure $\lambda_{0,x}$ on the space $D_{0,x}([0, \infty) \to \boldsymbol{R}^d)$ of the right-continuous paths having the left-hand limits and satisfying $X(0) = x$ whose characteristic function is given by

(2.1) $$\exp\{-t[(c^2p^2 + m^2c^4)^{1/2} - mc^2]\} = \int e^{ip(X(t)-X(0))} d\lambda_{0,x}(X).$$

The Lévy-Khinchin formula turns out to be

(2.2) $$(c^2p^2 + m^2c^4)^{1/2} - mc^2 = -\int_{\boldsymbol{R}^d \setminus \{0\}} [e^{ipy} - 1 - ipy I_{\{|y|<1\}}(y)]\, n(dy).$$

Here $n(dy)$ is the Lévy measure which is a σ-finite measure on $\boldsymbol{R}^d \setminus \{0\}$ satis-

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fying $\int_{R^d\setminus\{0\}}[y^2/(1+y^2)]n(dy)<\infty$. Notice that for $t>0$, (2.1) is a function of positive type in p.

For each path X in $D_{0,x}([0,\infty)\to\boldsymbol{R}^d)$, $N_X(ds\,dy)$ is a counting measure on $(0,\infty)\times(\boldsymbol{R}^d\setminus\{0\})$ for the associated stationary Poisson point process $X(s)-X(s-)$, $s\in\boldsymbol{D}_X\equiv\{s>0;\ X(s)\neq X(s-)\}$, on $\boldsymbol{R}^d\setminus\{0\}$:

$$N_X((t,t']\times U)=\sharp\{s\in\boldsymbol{D}_X;\ t<s\leqq t',\ X(s)-X(s-)\in U\},$$

where $0<t<t'$ and U is a Borel set in $\boldsymbol{R}^d\setminus\{0\}$. $\tilde{N}_X(ds\,dy)$ is defined as $\tilde{N}_X(ds\,dy)=N_X(ds\,dy)-\hat{N}(ds\,dy)$ with $\hat{N}(ds\,dy)\equiv\int N_X(ds\,dy)\,d\lambda_{0,x}(X)=ds\,n(dy)$. (In [3], N_X, $\tilde{N}_X$ and $\hat{N}$ are denoted by N_p, $\tilde{N}_p$ and $\hat{N}_p$.)

Now we assume that A is in $\mathscr{B}(\boldsymbol{R}^d\to\boldsymbol{R}^d)$ and Φ in $\mathscr{B}(\boldsymbol{R}^d\to\boldsymbol{R})$. Here $\mathscr{B}(\boldsymbol{R}^d\to\boldsymbol{R}^N)$, $N=1,d$, denotes the vector space of the $\boldsymbol{R}^N$-valued C^∞ functions in $\boldsymbol{R}^d$ which together with their derivatives of all orders are bounded in $\boldsymbol{R}^d$. It is shown that H defines a selfadjoint operator in $L^2(\boldsymbol{R}^d)$ with domain $H^1(\boldsymbol{R}^d)$ which is bounded from below.

The main result is the following theorem.

Theorem. *The solution $\psi(t,x)$ of the Cauchy problem for* (1.1) *with initial data $\psi(0,x)=g(x)$ in $L^2(\boldsymbol{R}^d)$ admits the following path integral representation*

$$\psi(t,x)=(e^{-t[H-mc^2]}g)(x)=\int e^{-S(t,0;X)}g(X(t))\,d\lambda_{0,x}(X) \tag{2.3}$$

with

$$\begin{aligned} S(t,0;X)=&\,i\int_0^{t+}\int_{|y|\geq 1}(e/c)A(X(s-)+y/2)\cdot yN_X(ds\,dy)\\ &+i\int_0^{t+}\int_{0<|y|<1}(e/c)A(X(s-)+y/2)\cdot y\tilde{N}_X(ds\,dy)\\ &+i\int_0^{t}\int_{0<|y|<1}(e/c)[A(X(s)+y/2)-A(X(s))]\cdot y\hat{N}(ds\,dy)\\ &+\int_0^t e\Phi(X(s))\,ds. \end{aligned} \tag{2.4}$$

3. Sketch of proof. Let $k_0(\tau,x)$ be the fundamental solution of the Cauchy problem for the free equation

$$\partial_t\psi(t,x)=-[(-c^2\Delta+m^2c^4)^{1/2}-mc^2]\psi(t,x),\quad t>0,\quad x\in\boldsymbol{R}^d, \tag{3.1}$$

to (1.1). Define a bounded linear operator $T(\tau)$, $\tau>0$, by

$$\begin{aligned}(T(\tau)g)(x)\\ =\int_{R^d}k_0(\tau,x-y)\exp\left[i(e/c)A\left(\frac{x+y}{2}\right)(x-y)-e\Phi\left(\frac{x+y}{2}\right)\tau\right]g(y)\,dy\end{aligned} \tag{3.2}$$

for $g\in L^2(\boldsymbol{R}^d)$. Then we have for $g\in L^2(\boldsymbol{R}^d)$,

$$\begin{aligned}(T(t/n)^n g)(x)=&\overbrace{\int_{R^d}\cdots\int_{R^d}}^{n}k_0(t/n,x^{(0)}-x^{(1)})\cdots k_0(t/n,x^{(n-1)}-x^{(n)})\\ &\cdot\exp[-S_n(x^{(0)},\dots,x^{(n)})]g(x^{(n)})\,dx^{(1)}\cdots dx^{(n)},\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}&S_n(x^{(0)},\dots,x^{(n)})\\ &=i\textstyle\sum_{j=1}^n(e/c)A\left(\frac{x^{(j-1)}+x^{(j)}}{2}\right)(x^{(j)}-x^{(j-1)})+\sum_{j=1}^n e\Phi\left(\frac{x^{(j-1)}+x^{(j)}}{2}\right)(t/n)\end{aligned} \tag{3.4}$$

with $x^{(0)}=x$. As $n\to\infty$, the left-hand side of (3.3) converges to $\exp[-t(H-mc^2)]g$ in L^2. The right-hand side of (3.3) is equal to the integral

$$(3.5)\qquad \int \exp[-S_n(X(t_0), X(t_1), \cdots, X(t_n))]g(X(t))d\lambda_{0,x}(X)$$

with respect to the measure $\lambda_{0,x}$ on the space $D_{0,x}([0, \infty)\to \boldsymbol{R}^d)$ where $t_j=jt/n$, and (3.5) approaches the last member of (2.3).

A full account of the present work will be published elsewhere.

References

[1] F. A. Berezin and M. A. Šubin: Symbols of operators and quantization. Coll. Math. Soc. Janos Bolyai, vol. 5, Hilbert Space Operators, Tihany (1970).

[2] A. Grossmann, G. Loupias and E. M. Stein: An algebra of pseudodifferential operators and quantum mechanics in phase space. Ann. Inst. Fourier (Grenoble), **18**, 343–368 (1968).

[3] N. Ikeda and S. Watanabe: Stochastic Differential Equations and Diffusion Processes. North-Holland/Kodansha, Amsterdam, Tokyo (1981).

[4] K. Itô: Stochastic Processes. Lecture Notes Series, vol. 16, Aårhus University (1969).

[5] L. D. Landau and E. M. Lifschitz: Course of Theoretical Physics. vol. 2, The Classical Theory of Fields, 4th revised English ed., Pergamon Press, Oxford (1975).

[6] M. M. Mizrahi: The Weyl correspondence and path integrals. J. Math. Phys., **16**, 2201–2206 (1975).

[7] ——: Phase space path integrals, without limiting procedure. ibid., **19**, 298–307 (1978); Erratum, ibid., **21**, 1965 (1980).

[8] B. Simon: Functional Integration and Quantum Physics. Academic Press, New York (1979).