26. On the Spectral Manifolds of the Simple Unilateral Shift and its Adjoint

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Summary. The concept of the "spectral manifold" is introduced by M. Radjabalipour [4] as a generalization or a modification of the spectral maximal space. In this paper, we show that the spectral manifolds of the simple unilateral shift S_0 on H^2 are only $\{0\}$ and H^2. And also we investigate some properties of the spectral manifolds of S_0^*.

1. Preliminaries. For a bounded linear operator T on the complex Banach space X, let

$$\sigma_p^0(T)=\{\lambda\in \boldsymbol{C}: (\omega I-T)f(\omega)\equiv 0 \text{ for some non-zero analytic function } f;\ D_r(\lambda)\to X\},$$

where $D_r(\lambda)=\{\omega\in \boldsymbol{C}: |\omega-\lambda|<r\}$ for some $r>0$. In case where $\sigma_p^0(T)=\phi$, T is said to have the single-valued extension property. For a closed set $\sigma\subset \boldsymbol{C}$, let

$$X_T(\sigma)=\{x\in X: (\omega I-T)f(\omega)\equiv x \text{ for some analytic function } f;\ \boldsymbol{C}\backslash\sigma\to X\},$$

and let $X_T(\tau)=\cup\{X_T(\sigma): \sigma\subset\tau \text{ and } \sigma \text{ is closed}\}$ for an arbitrary set $\tau\subset \boldsymbol{C}$. The set $X_T(\tau)$ is called the spectral manifold of T.

The following proposition is immediate.

Proposition 1.

(i) *$X_T(\tau)$ is a hyper-invariant* (i.e., *invariant under every operator which commutes with T*) *linear manifold of T.*

(ii) *If $\tau_1\subset\tau_2$, then $X_T(\tau_1)\subset X_T(\tau_2)$.*

(iii) *$X_T(\tau)=X_T(\tau\cap\sigma(T))$, $X_T(\sigma(T))=X$ and $X_T(\phi)=\{0\}$.*

(iv) *$X_T(\sigma)\subset\bigcap_{\omega\notin\sigma}(T-\omega I)X$ for any closed set $\sigma\subset \boldsymbol{C}$.*

Proposition 2. *If $X_T(\tau)$ is closed, then we have $\sigma(T|X_T(\tau))\subset\tau\cup\sigma_p^0(T)^{\sim}$ where "$\sim$" denotes the closure.*

Proof. If $x\in X_T(\tau)$, then $x\in X_T(\sigma)$ for some closed set $\sigma\subset\tau$ and $(\omega I-T)f(\omega)\equiv x$ for some analytic function f; $\boldsymbol{C}\backslash\sigma\to X$. Since $f(\omega)\in X_T(\sigma)$ for any $\omega\in \boldsymbol{C}\backslash\sigma$ by [1], $x\equiv(\omega I-T)f(\omega)\in(\omega I-T)X_T(\sigma)\subset(\omega I-T)X_T(\tau)$ for any $\omega\in \boldsymbol{C}\backslash\sigma\supset \boldsymbol{C}\backslash\tau$. By the assumption and by Proposition 1, $X_T(\tau)$ is a closed invariant subspace of T and hence $X_T(\tau)=(\omega I-T|X_T(\tau))X_T(\tau)$ for any $\omega\in \boldsymbol{C}\backslash\tau$. Next, if $(\lambda I-T)x=0$ for any $\lambda\in \boldsymbol{C}\backslash[\tau\cup\sigma_p^0(T)^{\sim}]$ and for some $x\in X_T(\tau)$, then $x\in X_T(\sigma)$ for some closed set $\sigma\subset\tau$ and hence $(\omega I-T)f(\omega)\equiv x$ for some analytic function f; $\boldsymbol{C}\backslash\sigma\to X$. Since $(\omega I-T)[f(\omega)-(\omega-\lambda)^{-1}x]=(\omega I-T)f(\omega)-(\omega I-\lambda I+\lambda I-T)(\omega-\lambda)^{-1}x=x-x-(\omega-\lambda)^{-1}(\lambda I-T)x=0$ on $\boldsymbol{C}\backslash[\sigma\cup\{\lambda\}]$, $f(\omega)$

$=(\omega-\lambda)^{-1}x$ on $D_r(\omega_0)\subset \boldsymbol{C}\backslash[\sigma\cup\{\lambda\}\subset\sigma_p^o(T)^{\sim}]$ and hence, on $\boldsymbol{C}\backslash[\sigma\cup\{\lambda\}\cup\sigma_p^o(T)^{\sim}]$ by the unicity theorem. Since $f(\omega)$ is analytic in $\boldsymbol{C}\backslash\sigma\supset\boldsymbol{C}\backslash[\sigma\cup\sigma_p^o(T)^{\sim}]\supset\boldsymbol{C}\backslash[\tau\cup\sigma_p^o(T)^{\sim}]\ni\lambda$, $x=0$. Hence $\lambda I-T|X_T(\tau)$ is injective for any $\lambda\in\boldsymbol{C}\backslash[\tau\cup\sigma_p^o(T)^{\sim}]\subset\boldsymbol{C}\backslash\tau$. Therefore $\lambda I-T|X_T(\tau)$ is invertible for any $\lambda\in\boldsymbol{C}\backslash[\tau\cup\sigma_p^o(T)^{\sim}]$ and hence $\sigma(T|X_T(\tau))\subset\tau\cup\sigma_p^o(T)^{\sim}$.

For convenience' sake, we state here the following known results.

Proposition 3 ([2]). *If $X_T(\bar{\tau})$ is closed for every open set $\tau\subset\boldsymbol{C}$, then $\sigma_p^o(T)=\phi$.*

Proposition 4 ([3]). *If T is hyponormal* (i.e., $T^*T\geqq TT^*$), *then $X_T(\sigma)$ is closed for all closed set $\sigma\subset\boldsymbol{C}$.*

As an immediate corollary we have

Corollary 1. *If T is hyponormal, then $\sigma_p^o(T)=\phi$.*

2. Main results. Throughout this section, we denote by S_0, the simple unilateral shift on H^2. It is known that $\sigma_p(S_0^*)=D_1(0)$ and $\sigma(S_0)=D_1(0)^{\sim}$.

Theorem 1. *If $[\boldsymbol{C}\backslash\tau]\cap D_1(0)^{\sim}\neq\phi$, then $X_{S_0}(\tau)=\{0\}$.*

Proof. Since $X_{S_0}(\sigma)$ is closed for a closed set $\sigma\subset\tau$ by Proposition 4, $\sigma(S_0|X_{S_0}(\sigma))\subset\sigma$ by Proposition 2 and Corollary 1. If $X_{S_0}(\sigma)\neq\{0\}$, then $S_0|X_{S_0}(\sigma)$ is a completely non-unitary isometry because S_0 is so. And hence we have $\sigma(S_0|X_{S_0}(\sigma))=D_1(0)^{\sim}$ and $D_1(0)^{\sim}\subset\sigma\subset\tau$ which contradicts with the assumption. Therefore $X_{S_0}(\sigma)=\{0\}$ for any closed set $\sigma\subset\tau$ and hence $X_{S_0}(\tau)=\{0\}$.

Above theorem means that the spectral manifolds of S_0 are only $\{0\}$ and H^2. And next, we shall investigate the spectral manifolds of the backward shift S_0^*.

Theorem 2. $X_{S_0^*}(\{\omega\in\boldsymbol{C}:\ |\omega|=1\})=H^2$.

Proof. For any $x\in H^2$ and $\omega\in D_1(0)$, let $f(\omega)=(I-\omega S_0)^{-1}x$, then f; $D_1(0)\to H^2$ is analytic and

$$S_0^*f(\omega)=S_0^*\sum_{n=0}^{\infty}\omega^nS_0^nx=S_0^*x+\omega\sum_{n=0}^{\infty}\omega^nS_0^nx=S_0^*x+\omega f(\omega)$$

and hence $(\omega I-S_0^*)f(\omega)\equiv -S_0^*x$. Therefore $S_0^*x\in X_{S_0^*}(\boldsymbol{C}\backslash D_1(0))=X_{S_0^*}([\boldsymbol{C}\backslash D_1(0)]\cap\sigma(S_0^*))=X_{S_0^*}(\{\omega\in\boldsymbol{C}:\ |\omega|=1\})$ by Proposition 1 (iii). And hence $S_0^*H^2\subset X_{S_0^*}(\{\omega\in\boldsymbol{C}:|\omega|=1\})$. And since $H^2=S_0^*S_0H^2\subset S_0^*H^2$, we have $H^2\subset X_{S_0^*}(\{\omega\in\boldsymbol{C}:|\omega|=1\})$.

In the proof of Theorem 2, let $x=e_0$, $e_0(z)\equiv 1$, then $f(\omega)$ is non-zero and $(\omega I-S_0^*)f(\omega)\equiv 0$ because $S_0^*e_0=0$, and hence $D_1(0)\subset\sigma_p^o(S_0^*)$. On the other hand, $\sigma_p^o(S_0^*)\subset\sigma_p(S_0^*)=D_1(0)$ and hence we have

Corollary 2. $\sigma_p^o(S_0^*)=D_1(0)$.

For an $\alpha\in D_1(0)$, let $f_\alpha(z)=(1-|\alpha|^2)^{1/2}(1-\alpha z)^{-1}$, then $f_\alpha(z)\in H^2$ and $S_0^*f_\alpha(z)=\alpha f_\alpha(z)$ and $S_\alpha=(S_0-\bar{\alpha}I)(I-\alpha S_0)^{-1}$ is the simple unilateral shift with its wandering unit vector $f_\alpha(z)$.

Theorem 3. *$S_\alpha^nf_\alpha(z)\in X_{S_0^*}(\{\alpha\})$ for all $n=0,1,2,\cdots$ and hence $X_{S_0^*}(\{\alpha\})$ is dense in H^2 for each $\alpha\in D_1(0)$.*

Proof. Since

$$(\omega I-S_0^*)(1-|\alpha|^2)(I-\bar{\alpha}S_0^*)^{-1}\left\{\frac{1}{\omega-\alpha}S_\alpha^n+\frac{1-\bar{\alpha}\omega}{(\omega-\alpha)^2}S_\alpha^{n-1}+\cdots+\frac{(1-\bar{\alpha}\omega)^n}{(\omega-\alpha)^{n+1}}I\right\}f_\alpha(z)$$

$$=(\omega I-S_0^*)\frac{1-|\alpha|^2}{1-\bar{\alpha}\omega}(I-\bar{\alpha}S_0^*)^{-1}\left\{\frac{1-\bar{\alpha}\omega}{\omega-\alpha}S_\alpha^n+\left(\frac{1-\bar{\alpha}\omega}{\omega-\alpha}\right)^2S_\alpha^{n-1}+\cdots+\left(\frac{1-\bar{\alpha}\omega}{\omega-\alpha}\right)^{n+1}I\right\}f_\alpha(z)$$

$$=\left(\frac{u+\alpha}{1+\bar{\alpha}u}I-S_0^*\right)(1+\bar{\alpha}u)(I-\bar{\alpha}S_0^*)^{-1}\left\{\frac{1}{u}S_\alpha^n+\frac{1}{u^2}S_\alpha^{n-1}+\cdots+\frac{1}{u^{n+1}}I\right\}f_\alpha(z),$$

where $u=\dfrac{\omega-\alpha}{1-\bar{\alpha}\omega}$

$$=\{(u+\alpha)I-(1+\bar{\alpha}u)S_0^*\}(I-\bar{\alpha}S_0^*)^{-1}\frac{1}{u^{n+1}}\{u^nI+u^{n-1}S_\alpha^*+\cdots+S_\alpha^{*n}\}S_\alpha^nf_\alpha(z)$$

$$=\{u(I-\bar{\alpha}S_0^*)-(S_0^*-\alpha I)\}(I-\bar{\alpha}S_0^*)^{-1}\frac{1}{u^{n+1}}\{u^nI+u^{n-1}S_\alpha^*+\cdots+S_\alpha^{*n}\}S_\alpha^nf_\alpha(z)$$

$$=(uI-S_\alpha^*)\frac{1}{u^{n+1}}\{u^nI+u^{n-1}S_\alpha^*+\cdots+S_\alpha^{*n}\}S_\alpha^nf_\alpha(z)$$

$$=\frac{1}{u^{n+1}}(u^{n+1}I-S_\alpha^{*n+1})S_\alpha^nf_\alpha(z)=\frac{1}{u^{n+1}}\{u^{n+1}S_\alpha^nf_\alpha(z)-S_\alpha^*f_\alpha(z)\}=S_\alpha^nf_\alpha(z)$$

for all $\omega\in C\backslash\{\alpha\}$ and since

$$g(\omega)=(1-|\alpha|^2)(I-\bar{\alpha}S_0^*)^{-1}\left\{\frac{1}{\omega-\alpha}S_\alpha^n+\frac{1-\bar{\alpha}\omega}{(\omega-\alpha)^2}S_\alpha^{n-1}+\cdots+\frac{(1-\bar{\alpha}\omega)^n}{(\omega-\alpha)^{n+1}}I\right\}f_\alpha(z)$$

is an H^2-valued, analytic function on $C\backslash\{\alpha\}$, $S_\alpha^nf_\alpha(z)\in X_{S_0^*}(\{\alpha\})$. Since $X_{S_0^*}(\{\alpha\})$ is a linear manifold by Proposition 1 (i) and since $\{S_\alpha^nf_\alpha(z);\ n=0, 1, 2, \cdots\}$ is a complete orthonormal basis of H^2, $X_{S_0^*}(\{\alpha\})$ is dense in H^2.

Corollary 3. *$S_\alpha^nf_\alpha(z)\in\bigcap_{\omega\in C}(S_0^*-\omega I)H^2$ for all $\alpha\in D_1(0)$ and all $n=0, 1, 2, \cdots$.*

Proof. $S_\alpha^nf_\alpha(z)\in X_{S_0^*}(\{\alpha\})$, by Theorem 3, $\subset\bigcap_{\omega\notin\{\alpha\}}(S_0^*-\omega I)H^2$, by Proposition 1 (iv), $=\bigcap_{\omega\in C}(S_0^*-\omega I)H^2$, because $H^2=S_\alpha^*S_\alpha H^2\subset S_\alpha^*H^2=(S_0^*-\alpha I)H^2$.

Theorem 4. *If $[C\backslash\tau]\cap\{\omega\in C:\ |\omega|=1\}\neq\phi$, then $f_\alpha(z)\notin X_{S_0^*}(\tau)$ for all $\alpha\in D_1(0)\backslash\tau$.*

Proof. If $f_\alpha(z)\in X_{S_0^*}(\tau)$ for some $\alpha\in D_1(0)\backslash\tau$, then $f_\alpha(z)\in X_{S_0^*}(\sigma)$ for some closed set $\sigma\subset\tau$ and then $(\omega I-S_0^*)g(\omega)\equiv f_\alpha(z)$ for some analytic function g; $C\backslash\sigma\to H^2$ and $0=(\alpha I-S_0^*)f_\alpha(z)=(\alpha I-S_0^*)(\omega I-S_0^*)g(\omega)=(\omega I-S_0^*)(\alpha I-S_0^*)g(\omega)$ and $(\alpha I-S_0^*)g(\omega)$ is an H^2-valued, analytic function on $C\backslash\sigma$. If $(\alpha I-S_0^*)g(\omega)\neq0$, then $C\backslash\tau\subset C\backslash\sigma\subset\sigma_p^o(S_0^*)=D_1(0)$ by Corollary 2 and this contradicts with the assumption. Hence $(\alpha I-S_0^*)g(\omega)=0$. Since S_α is the simple unilateral shift with $\{C\cdot f_\alpha(z)\}$ as the null space of its adjoint, $g(\omega)=h(\omega)f_\alpha(z)$ for some scalar-valued, analytic function $h(\omega)$ on $C\backslash\sigma$. And hence

$$f_\alpha(z)\equiv(\omega I-S_0^*)g(\omega)=(\omega I-S_0^*)h(\omega)f_\alpha(z)=h(\omega)(\omega I-S_0^*)f_\alpha(z)$$
$$=h(\omega)(\omega-\alpha)f_\alpha(z).$$

Therefore $h(\omega)(\omega-\alpha)\equiv1$ on $C\backslash\sigma$ and this is a contradiction because $\alpha\in D_1(0)\backslash\tau\subset C\backslash\sigma$.

Remark. If $\phi \neq \tau \subsetneqq \{\omega \in \boldsymbol{C} : |\omega|=1\}$, then $f_\alpha(z) \notin X_{S_0^*}(\tau)$ for all $\alpha \in D_1(0)$ by Theorem 4. Since $X_{S_0^*}(\tau)$ is invariant under S_α^* by Proposition 1 (i), $S_\alpha^n f_\alpha(z) \notin X_{S_0^*}(\tau)$ for all $n=0, 1, 2, \cdots$ and hence $H^2 \backslash X_{S_0^*}(\tau)$ is dense in H^2. But, in this case, it is an open question whether $X_{S_0^*}(\tau)$ is dense in H^2 or not.

References

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