# 25. On a Multi-dimensional Inverse Parabolic Problem

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**1. Introduction.** Inspired by the Gel'fand-Levitan theory [1], we have studied certain evolutional inverse problems of one space dimension ([3–6]). The purpose of the present article is to extend the related work [2] to a multi-dimensional case. Although our problem is special, our method would apply to more general ones.

For $I=(0,1)$ and $S^1=\{e^{i2\pi\theta}\,|\,0\leqq\theta<1\}$, let $\Omega$ be $I\times S^1$. Then, $\partial\Omega=\gamma_0\cup\gamma_1$, where $\gamma_0=\{0\}\times S^1$ and $\gamma_1=\{1\}\times S^1$. For $p\in C^\infty(\bar{\Omega})$ and $F\in C^\infty(\partial\Omega\times[0,T_1])$, we consider the parabolic equation

$$\frac{\partial u}{\partial t}=\Delta u-pu \qquad (z=(x,\theta)\in\Omega,\ 0\leqq t\leqq T_1) \tag{1}$$

with

$$\left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}=F \qquad (0\leqq t\leqq T_1) \tag{2}$$

and

$$u|_{t=0}=0 \qquad (z\in\Omega). \tag{3}$$

Here $\Delta=(\partial^2/\partial x^2)+(\partial^2/\partial\theta^2)$, $\nu$ denotes the outer unit normal vector on $\partial\Omega$ and $T_1>0$. The problem we study is to determine $p$ through $F\neq0$ and $f=u|_{\partial\Omega}$ $(0\leqq t\leqq T_1)$.

Henceforth, $u=u(z,t;p,F)$ denotes the solution of (1) with (2) and (3). $A_p$ is the differential operator $-\Delta+p$ with the Neumann boundary condition $(\partial/\partial\nu)|_{\partial\Omega}=0$. $\sigma(A_p)=\{\lambda_i\}_{i=0}^\infty$ $(-\infty<\lambda_0\leqq\lambda_1\leqq\cdots\to\infty)$ denote its eigenvalues and $\phi_i$ $(\|\phi_i\|_{L^2(\Omega)}=1)$ is its eigenfunction corresponding to $\lambda_i$. For simplicity, each $\lambda_i$ is supposed to be simple: $-\infty<\lambda_0<\lambda_1<\cdots\to\infty$. Then we have

**Theorem 1.** *Suppose that for $F=g(t)h(\xi)$ $(0\leqq t\leqq T_1,\ \xi\in\partial\Omega)$ satisfying $g\neq0$ and*

$$\int_{\partial\Omega}h(\xi)\phi_i(\xi)d\sigma_\xi\neq0 \qquad (i=0,1,\cdots), \tag{4}$$

*the relation*

$$u(\xi,t;q,F)=u(\xi,t;p,F) \qquad (\xi\in\partial\Omega,\ 0\leqq t\leqq T_1) \tag{5}$$

*holds for some coefficient $q$. Then the equality*

$$q\equiv p \tag{6}$$

*follows, provided that $p$ and $q$ are real analytic.*

**2. Outline of the proof of Theorem 1.** The solution $u=u(z,t;p,F)$ of (1) with (2) and (3) is given as

$$u=u(z,t)=\int_0^t d\tau\int_{\partial\Omega}d\sigma_\xi G(z,\xi;t-\tau;p)F(\tau,\xi),$$

where $G$ is the Green function of $-(\partial/\partial t)+A_p: G(z, w; t; p)=\sum_{i=0}^{\infty} e^{-t\lambda_i}\cdot$ $\phi_i(z)\phi_i(w)$. Since $F(t, \xi)=g(t)h(\xi)$, we have

$$u(z, t; p, F)=\int_0^t r(z, t-\tau)g(\tau)d\tau,$$

where

$$r(z, t)=\sum_{i=0}^{\infty} e^{-t\lambda_i}\phi_i(z)\int_{\partial\Omega}\phi_i(\xi)h(\xi)d\sigma_\xi. \tag{7}$$

Similarly, the relation

$$u(z, t; q, F)=\int_0^t s(z, t-\tau)g(\tau)d\tau$$

holds with

$$s(z, t)=\sum_{i=0}^{\infty} e^{-t\mu_i}\psi_i(z)\int_{\partial\Omega}\psi_i(\xi)h(\xi)d\sigma_\xi, \tag{8}$$

where $\{\mu_i\}_{i=0}^{\infty}$ $(-\infty<\mu_0\leqq\mu_1\leqq\cdots\to\infty)$ and $\{\psi_i\}_{i=0}^{\infty}$ $(\|\psi_i\|_{L^2(\Omega)}=1)$ denote the eigenvalues and the eigenfunctions of $A_q$, respectively. From the assumption (5), we have

$$\int_0^t \{r(\xi, t-\tau)-s(\xi, t-\tau)\}g(\tau)d\tau=0 \qquad (\xi\in\partial\Omega,\ 0\leqq t\leqq T_1),$$

hence

$$r(\xi, t)=s(\xi, t) \qquad (\xi\in\partial\Omega,\ 0\leqq t\leqq T_1) \tag{9}$$

because of $g\neq 0$. By the analyticity in $t$ of $r$ and $s$, the equality (9) holds for $0\leqq t<\infty$. We compare the behaviors as $t\to\infty$ of both sides of (10). By virtue of Weyl's formula, the assumption (4), and the fact $\phi_i|_{\partial\Omega}\neq 0$, we can show that each $\mu_i$ is simple, $\lambda_i=\mu_i$, and

$$\phi_i(\xi)\int_{\partial\Omega}\phi_i(\eta)h(\eta)d\sigma_\eta=\psi_i(\xi)\int_{\partial\Omega}\psi_i(\eta)h(\eta)d\sigma_\eta \qquad (\xi\in\partial\Omega,\ i=0, 1, \cdots).$$

The last equalities imply $\phi_i(z)=c_i\psi(z)$ $(z\in\partial\Omega)$ with $c_i^2=1$, and Theorem 1 is reduced to the following

**Theorem 2.** *The relation*

$$\lambda_i=\mu_i \quad \text{and} \quad \phi_i|_{\partial\Omega}=c_i\psi_i|_{\partial\Omega} \quad (i=0, 1, 2, \cdots) \tag{10}$$

*with* $c_i^2=1$ *imply* $q\equiv p$, *if* $p$ *and* $q$ *are real analytic.*

**3. Outline of the proof of Theorem 2.** For sufficiently large $\lambda>0$ and $s>0$.

$$K_s(z, w; \lambda)=\sum_{i=0}^{\infty}\{c_i\psi_i(z)-\phi_i(z)\}\phi_i(w)\ (\lambda_i+\lambda)^{-s}$$

becomes a $C^2$-function of $(z, w)\in\bar{\Omega}\times\bar{\Omega}$. Putting $\square=-\Delta_z+\Delta_w$ and $c(z, w)=-q(z)+p(w)$, we have

$$(\square-c(z, w))K_s(z, w; \lambda)=c(z, z)G_s(z, w; p, \lambda)$$

from the first relation of (10), where $G_s(z, w; p, \lambda)=\sum_{i=0}^{\infty}\phi_i(z)\phi_i(w)\ (\lambda_i+\lambda)^{-s}$ is the Green function of $(A_p+\lambda)^s$. On the other hand, the equality

$$K_s|_{\Gamma_1}=\frac{\partial}{\partial\nu}K_s|_{\Gamma_1}=0$$

follows from the second equalities of (10), where $\Gamma_1=\gamma_0\times\partial\Omega\subset\partial(\Omega\times\Omega)$ and $\nu$ is the outer unit normal vector on $\Gamma_1$. Set $D=\{(z, z)\,|\,z\in\Omega\}\subset\Omega\times\Omega$. Then, $G_s(\cdot, \cdot; p, \lambda)$ is real analytic in $\bar{\Omega}\times\bar{\Omega}\backslash\bar{D}$. Furthermore, $\Gamma_1$ is noncharacteristic with respect to $\square$. Therefore, by Cauchy-Kowalevskaja's theorem and Holmgren's one, $K_s(\cdot, \cdot; \lambda)$ is real analytic in a neighborhood $U_1$ of $\Gamma_1$ in $\Omega\times\Omega\backslash\bar{D}$. Actually, $U_1$ can contain all points in $\Omega\times\Omega\backslash\bar{D}$ which are reached by deforming a portion of the initial surface $\gamma_1$ analytically through noncharacteristic surfaces with respect to $\square$ having the same boundary. We note that in the $x$-$y$ plane, there is an analytic family of noncharacteristic curves $\{C_\lambda\}_{0\leqq\lambda<1}$ with respect to $(\partial^2/\partial x^2)-(\partial^2/\partial y^2)$ such that $C_0=\{x=0,\ y\in\bar{I}\}$, $\partial C_\lambda=\partial C_0=\{(0,0),(1,1)\}$, and $\bigcup_{0\leqq\lambda<1}C_\lambda=\{(x,y)\,|\,0\leqq x<1/2,\ x<y<1-x\}$. Then, the family $\{\tilde{C}_\lambda\}_{0\leqq\lambda<1}$ defined by $\tilde{C}_\lambda=\{(x,\theta,y,\omega)\,|\,(x,y)\in C_\lambda,\ \theta\in S^1,\ \omega\in S^1\}$ satisfies the condition given above. Consequently, we can take $U_1=\{(x,\theta,y,\omega)\,|\,0\leqq x<1/2,\ x<y<1-x,\ \theta\in S^1,\ \omega\in S^1\}$. Therefore,

$$K=K(z,w)=(-\Delta_w+p(w)+\lambda)^s K_s(z,w;\lambda)\in\mathcal{D}'(\Omega\times\Omega) \tag{11}$$

is real analytic in $U_1$ and satisfies

$$(\square-c(z,w))K=c(z,z)\delta(z-w) \tag{12}$$

in $\Omega\times\Omega$ with $K|_{\Gamma_1}=(\partial/\partial\nu)K|_{\Gamma_1}=0$. Again by Holmgren's theorem, we obtain $K=0$ in $U_1\subset\bar{\Omega}\times\bar{\Omega}\backslash D$. We now recall $c_i^2=1$ and consider the function

$$F_s(z,w;\lambda)=\sum_{i=0}^{\infty}\psi_i(z)\{c_i\phi_i(w)-\psi_i(w)\}\ (\lambda_i+\lambda)^{-s}.$$

By the same argument for $\Gamma_2=\Omega\times\gamma_0$, $F_s$ is shown to be real analytic in $U_2=\{(x,\theta,y,\omega)\,|\,0\leqq y<1/2,\ y<x<1-y,\ \theta\in S^1,\ \omega\in S^1\}$, and the distribution $F=F(z,w)=(-\Delta_z+q(z)+\lambda)^s F_s(z,w;\lambda)$ becomes zero in $U_2$. However, we can show that $F=K$ by a standard argument. In particular $K=0$ in $U_1\cup U_2=\{(x,\theta,y,\omega)\,|\,x+y<1;\ 0\leqq x,y;\ \theta,\omega\in S^1;\ x\neq y\}$. We may regard $K=K(z,\cdot)$ as a $w^*-C^2$ function of $z$ in $\mathcal{D}'(\Omega)$. Then, the same argument for $\gamma_1$ implies

$$\operatorname{supp}K(z,\cdot)\subset\{y=x\}\cup\{y=1-x\}. \tag{13}$$

Therefore, we have

$$K(z,w)=\sum_{l=0}^{m}a_l(z,\omega)\otimes\delta^{(l)}(x-y)+\sum_{l=0}^{n}b_l(z,\omega)\otimes\delta^{(l)}(1-x-y),$$

$a_l(z,\cdot)$, $b_l(z,\cdot)\in\mathcal{D}'(S^1)$ being $w^*-C^2$ in $z$. Substituting this equality into (12), we get

$$\frac{\partial}{\partial x}a_m(z,\omega)=\frac{\partial}{\partial x}b_n(z,\omega)=0. \tag{14}$$

On the other hand, we obtain

$$\begin{aligned}c_i\psi_i(z)=\phi_i(z)&+\textstyle\sum_{l=0}^{m}{}_{\mathcal{D}'(S^1)}\left\langle a_l(z,\cdot),\ \dfrac{\partial^l}{\partial x^l}\phi_i(x,\cdot)\right\rangle_{\mathcal{D}(S^1)}\\&+\textstyle\sum_{l=0}^{n}{}_{\mathcal{D}'(S^1)}\left\langle b_l(z,\cdot),\ \dfrac{\partial^l}{\partial x^l}\phi_i(1-x,\cdot)\right\rangle_{\mathcal{D}(S^1)},\end{aligned}$$

so that

$$\begin{aligned}0&=\left\{\sum_{l=0}^{m}\left\langle a_l(z,\cdot),\ \frac{\partial^l}{\partial x^l}\phi_i(x,\cdot)\right\rangle+\sum_{l=0}^{n}\left\langle b_l(z,\cdot),\ \frac{\partial^l}{\partial x^l}\phi_i(1-x,\cdot)\right\rangle\right\}\Bigg|_{x=0,1}\\&=\frac{\partial}{\partial x}\left\{\sum_{l=0}^{m}\left\langle a_l(z,\cdot),\ \frac{\partial^l}{\partial x^l}\phi_i(x,\cdot)\right\rangle+\sum_{l=0}^{n}\left\langle b_l(z,\cdot),\ \frac{\partial^l}{\partial x^l}\phi_i(1-x,\cdot)\right\rangle\right\}\Bigg|_{x=0,1}\end{aligned} \tag{15}$$

for $i=0,1,2,\cdots$, by (10). We can show that the relation (14)–(15) implies

$a_m=b_n=0$, hence $a_l=0$ $(0\leqq l\leqq m)$ and $b_l=0$ $(0\leqq l\leqq n)$ by an induction. Thus $K\equiv 0$ holds, and $q\equiv p$ follows from (12).

## References

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